PHY/V/CC/09 **Student's Copy**

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(CBCS)

(5th Semester)

PHYSICS

FIFTH PAPER

(Mathematical Physics—II)

Full Marks : 75

Time : 3 hours

(PART : A—OBJECTIVE)

(*Marks* : 25)

The figures in the margin indicate full marks for the questions

SECTION—A

(*Marks* : 10)

Tick (✓) the correct answer in the brackets provided : 1×10=10

1. The analytic function $f(z)$ whose real part is $x^2 \quad y^2$ is

(a) z ( ) (b) z^2 ( )

(c) $z^{\;2}$ ( ) (d) z^3 ( )

2. The value of the integral

$$I \quad \frac{1}{2\;i} \circ_C \frac{dz}{z\quad 3}$$

where C is the circle $|z| \quad 1$ is

(a) 0 ( ) (b) 0·5 ( )

(c) 1 ( ) (d) 2 ( )

3. If a function of two variables is a solution of Laplace's equation, then the function is said to be

(a) conjugate ( )

(b) anharmonic ( )

(c) harmonic ( )

(d) analytic ( )

4. For the linear differential equation

$$y \quad \frac{1}{x \quad 1} y \quad \frac{x}{(x \quad 1)^2} y \quad 0$$

where $y \quad \frac{dy}{dx}$

(a) $x \quad 0$ is a singular point ( )

(b) $x \quad 1$ is a singular point ( )

(c) $x \quad 1$ is an ordinary point ( )

(d) $x \quad 1$ is a singular point ( )

5. If the Legendre polynomial

$$P_5(x) \quad k \quad x^5 \quad \frac{70}{63} x^3 \quad \frac{15}{63} x$$

then k is equal to

(a) $\frac{63}{2}$ ( )

(b) $\frac{21}{2}$ ( )

(c) $\frac{63}{5}$ ( )

(d) $\frac{63}{8}$ ( )

6. The value of the integral $x J_0(x)\, dx$ is

(a) $x J_1(x) \quad J_0(x)$ ( )

(b) $xJ_1(x)$ ( )

(c) $J_1(x)$ ( )

(d) $x^2 J_n(x)$ ( )

7. The Fourier transform of $\frac{df}{dt}$, i.e., F T $\frac{df}{dt}$ is

(a) $\frac{}{\sqrt{2}}$ $f(t)e^{\ i\ \ t}dt$ ( )

(b) $\sqrt{\frac{}{2}}$ $f(t)e^{i\ \ t}dt$ ( )

(c) $\frac{i}{\sqrt{2}}$ $f(t)e^{\ i\ \ t}dt$ ( )

(d) $\frac{1}{i\ \sqrt{2}}$ $f(t)e^{i\ \ t}dt$ ( )

8. The Fourier series of a function $f(x)\ \ x\ \ x^2$ is given by

$$x\ \ x^2\ \ a_0\ \ _{n\ 1}a_n \cos nx\ \ _{n\ 1}b_n \sin nx$$

then the value of a_0 in the interval x is

(a) ^{2} ( ) (b) $\frac{2}{2}$ ( )

(c) $\frac{2}{2}$ ( ) (d) $2\frac{2}{3}$ ( )

9. If $f(s)$ is the Laplace transform of $F(t)$, then $\mathscr{L}\ ^1[f(as)]$ is

(a) $\frac{1}{a}F\ \frac{t}{a}$ ( ) (b) $\frac{1}{a}F\ \frac{a}{t}$ ( )

(c) $aF\ \frac{t}{a}$ ( ) (d) $aF\ \frac{a}{t}$ ( )

10. Laplace transform of $t\ ^{1/2}$ is

(a) $\frac{\sqrt{}}{s}$ ( ) (b) $\sqrt{\frac{}{s}}$ ( )

(c) $\frac{s}{\sqrt{}}$ ( ) (d) $\frac{1}{\sqrt{\ \ s}}$ ( )

SECTION—B

(*Marks* : 15)

Answer the following questions : 3×5=15

1. Test the analyticity of the function $f(z) = \frac{1}{z}$.

OR

2. Prove that $u = x^2 - y^2$ and $v = \frac{y}{x^2 + y^2}$ are harmonic functions of (x, y), but are not harmonic conjugates.

3. Explain the terms ordinary point and singular point of a second-order differential equation.

OR

4. Show that $u(x, t) = (A\cos \lambda x + B\sin \lambda x)e^{-\lambda^2 h^2 t}$ is the solution of one-dimensional heat flow equation $\frac{\partial u}{\partial t} = h^2 \frac{\partial^2 u}{\partial x^2}$, where A, B and λ are constants to be determined from boundary conditions.

5. Prove that $H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}$, where $H_n(x)$ is Hermite polynomial.

OR

6. Using Rodrigue's formula for Legendre's polynomials, show that $\int_{-1}^{1} P_0(x)\, dx = 2$.

7. If $g(\omega)$ is the Fourier transform of $f(t)$, then show that the Fourier transform of $f(t)\cos at$ is given by $\frac{1}{2}[g(\omega - a) + g(\omega + a)]$.

OR

8. If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ be the Fourier series for the function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ k, & 0 < x < \pi \end{cases}$$

then show that $b_n = \frac{k}{n\pi}[1 - (-1)^n]$.

9. Find the Laplace transform of sawtooth wave function

$$F(t) \quad \frac{at}{T} \text{ for } 0 \quad t \quad T \text{ and } F(t \quad T) \quad F(t)$$

OR

10. If $f(s)$ is the Laplace transform of $F(t)$, then show that $\mathscr{L}\ ^{1}[f(s \quad a)] \quad e^{at}F(t)$.

(PART : B—DESCRIPTIVE)

(*Marks* : 50)

The figures in the margin indicate full marks for the questions

1. *(a)* Derive the necessary conditions to be satisfied by the real and imaginary parts of a complex analytic function. 3

(b) Using Cauchy's residue theorem, show that

$$\begin{matrix} 2 \\ 0 \end{matrix} \frac{\cos 3}{5 \quad 4\cos} d \qquad \overline{12}$$ 5

(c) Find the residue of $\frac{z}{(z \quad a)(z \quad b)}$ at infinity. 2

OR

2. *(a)* State and prove Cauchy's integral theorem. 1+2=3

(b) Using Cauchy's integral theorem, integrate $\frac{z^2 \quad 1}{z^2 \quad 1}$ along a circle of radius 1 with centre at *(i)* $z \quad 1$ and *(ii)* $z \quad i$. 2+2=4

(c) Obtain the Taylor's expansion of the function $f(z) \quad \frac{2z^3 \quad 1}{z^2 \quad z}$ about $z \quad 1$. 3

3. *(a)* Obtain the power series solutions for the differential equation

$$(1 \quad z^2)\frac{d^2 f}{dz^2} \quad 2z\frac{df}{dz} \quad 2f \quad 0$$

about $z \quad 1$. 6

(b) Write down the Laplace equation in two-dimensional Cartesian coordinates and solve it by the method of separation of variables. 4

OR

4. *(a)* Solve the differential equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

under the boundary conditions $y(0, t) = 0$; $y(l, t) = 0$; $y(x, 0) = a \sin \frac{\pi x}{l}$ and $\frac{\partial y(x, 0)}{\partial t} = 0$. 6

(b) Find the series solution of $(1 - x^2) y'' - x y' + y = 0$ about $x = 0$. 4

5. *(a)* Show that $P_n(x)$ is coefficient of h^n in the expansion of $[1 - 2xh + h^2]^{-\frac{1}{2}}$ in ascending powers of h. Hence, show that *(i)* $P_n(1) = 1$ and *(ii)* $P_n(-x) = (-1)^n P_n(x)$. 4+3=7

(b) Using the expression

$$e^{x^2 - (z - x)^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n$$

show that $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2})$. 3

OR

6. *(a)* Using the expression

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \, \Gamma(n + r + 1)} \left(\frac{x}{2} \right)^{n + 2r}$$

prove the following relations : 5

(i) $2 J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

(ii) $2n J_n(x) = x \{ J_{n-1}(x) + J_{n+1}(x) \}$

(b) Using the expansion of Bessel's functions, show that

(i) $J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - x \sin\theta) \, d\theta$

(ii) $J_0(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x \sin\theta) \, d\theta$ 3+2=5

7. *(a)* Find the Fourier series expansion of the periodic function of period 2 defined as

$$f(x) \quad \begin{cases} k, & \quad x \quad 0 \\ k, & 0 \quad x \end{cases}$$

Hence show that

$$\frac{}{4} \quad 1 \quad \frac{1}{3} \quad \frac{1}{5} \quad \frac{1}{7} \quad \cdots$$ 5+1=6

(b) Find the finite Fourier sine transform and cosine transform of $f(x) \quad x$, $0 \quad x \quad 2$. 2+2=4

OR

8. *(a)* Obtain the Fourier series for a function of

$$f(x) \quad \begin{cases} 0, & \quad x \quad 0 \\ x, & 0 \quad x \end{cases}$$

Hence show that

(i) $\frac{}{4} \quad 1 \quad \frac{1}{3} \quad \frac{1}{5} \quad \frac{1}{7} \quad \cdots$

(ii) $\frac{^2}{8} \quad 1 \quad \frac{1}{3^2} \quad \frac{1}{5^2} \quad \frac{1}{7^2} \quad \cdots$ 4+2=6

(b) If $g(k)$ be the Fourier transform of $f(x)$, then show that the Fourier transform of the nth derivative of $f(x)$ is $(\ ik)n \, g(k)$. 4

9. *(a)* If $\mathscr{L}[J_0(t)] \quad \frac{1}{\sqrt{s^2 \quad 1}}$, then show that $\int_0^t J_0(u) J_0(t \quad u) \, du \quad \sin t$. 3

(b) Find the inverse Laplace transform of

$$f(s) \quad \log \frac{s^2 \quad 1}{s^2}$$ 3

(c) Find the Laplace transform of $F(t) \quad t^2 e^t \sin 4t$. 4

OR

10. *(a)* Using Laplace transform, show that $_0 \cos x^2 dx \quad \frac{1}{2}\sqrt{\frac{}{2}}$. 5

(b) Using convolution theorem, find the inverse Laplace transforms of $\frac{1}{s^2(s^2 \quad ^2)}$. 3

(c) Find the Laplace transform of the function $F(t) \quad \frac{e^{at} \quad 1}{a}$. 2

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