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(6th Semester)

MATHEMATICS

Paper : Math–362

(Advanced Calculus)

Full Marks : 75

Time : 3 hours

(PART : A—OBJECTIVE)

(*Marks* : 25)

The figures in the margin indicate full marks for the questions

SECTION—A

(*Marks* : 10)

Answer **all** questions

Each question carries 1 mark

Tick (✓) the correct answer in the brackets provided :

1. If P and S are any two partitions of $[a, b]$, then

(a) $L(P, f) \quad U(S, f)$ ( )

(b) $U(S, f) \quad U(P, f)$ ( )

(c) $U(S, f) \quad L(P, f)$ ( )

(d) $U(P, f) \quad U(S, f)$ ( )

2. If $f_1, f_2 \quad R[a, b]$, then which of the following is incorrect?

(a) $f_1^2 \quad R[a, b]$ ( )

(b) $f_1 \quad f_2 \quad R[a, b]$ ( )

(c) $f_1 / f_2 \quad R[a, b]$ ( )

(d) $f_2^2 \quad R[a, b]$ ( )

3. If f and g be two positive functions such that $f(x) \quad g(x)$, for all $x \quad [a, b]$, then the improper integral

(a) ${}_a^b g\, dx$ converges if ${}_a^b f\, dx$ diverges ( )

(b) ${}_a^b f\, dx$ converges if ${}_a^b g\, dx$ converges ( )

(c) ${}_a^b f\, dx$ diverges if ${}_a^b g\, dx$ diverges ( )

(d) Both *(b)* and *(c)* ( )

4. The definite integral ${}_0^2 \frac{1}{(x \quad 1)(x \quad 2)} dx$ is

(a) improper integral of first kind ( )

(b) improper integral of second kind ( )

(c) convergent at left end ( )

(d) convergent at both ends ( )

5. The improper integral $_{a} \frac{1}{x^n} dx$, $a \quad 0$ converges if and only if

(a) $n \quad 1$ ( )

(b) $n \quad 1$ ( )

(c) $n \quad 1$ ( )

(d) $n \quad 1$ ( )

6. The uniformly convergent improper integral of a continuous function

(a) is not continuous ( )

(b) is itself continuous ( )

(c) may be continuous ( )

(d) None of the above ( )

7. The value of $xy(x \quad y)\, dxdy$ over the area between $y \quad x^2 \quad 0$ and $y \quad x \quad 0$ is

(a) 56 ( )

(b) $\frac{3}{16}$ ( )

(c) $\frac{3}{46}$ ( )

(d) $\frac{3}{56}$ ( )

8. The sequence $\{f_n(x)\}$, where $f_n(x) \quad \frac{\sin(nx \quad n)}{n}$, $x \quad \mathscr{R}$, $n \quad 1, 2, 3, \cdots$ is

(a) divergent ( )

(b) convergent and converges to 0 ( )

(c) convergent and converges to 1 ( )

(d) convergent and converges to 2 ( )

9. With regard to uniform and pointwise convergence of sequences in $[a, b]$, which of the following is true?

(a) Pointwise convergence implies uniform convergence ( )

(b) Uniform convergence implies pointwise convergence ( )

(c) Uniform limit   pointwise limit ( )

(d) All of the above ( )

10. The series, $\dfrac{1}{nx \quad 1}$

(a) is uniformly convergent ( )

(b) converges to 0 but not uniformly ( )

(c) converges to 1 but not uniformly ( )

(d) is not integrable term-by-term ( )

SECTION—B

(*Marks* : 15)

Each question carries 3 marks

1. Show that

$$f(x) \quad \begin{matrix} 0, \text{ when } x \text{ is irrational} \\ 1, \text{ when } x \text{ is rational} \end{matrix}$$

is not Riemann integrable.

2. Examine the convergence of

$$_0^1 \sqrt{\frac{1 \quad x}{1 \quad x}}\, dx$$

3. Evaluate

$$_C \frac{ydx \quad xdy}{x^2 \quad y^2}$$

round the circle $C : x^2 \quad y^2 \quad 1$.

4. Define pointwise and uniform convergence of a sequence of real-valued functions.

5. Show that the sequence, $\{\sin(nx \quad n)/n\}$ for any real number x and natural number n, is convergent to zero.

(PART : B—DESCRIPTIVE)

(*Marks* : 50)

The figures in the margin indicate full marks for the questions

Answer **one** question from each Unit

UNIT—I

1. *(a)* State and prove Darboux theorem. 1+4=5

(b) Show that

$$f(x) \quad \begin{matrix} \dfrac{1}{2^n}, & \dfrac{1}{2^{n \quad 1}} \quad x \quad \dfrac{1}{2^n}, \; n \quad 0, 1, 2, \cdots \\ 0\,, & x \quad 0 \end{matrix}$$

is R-integrable on $[a, b]$ and find the value of $\,_0^1 f\,dx$. 5

2. *(a)* Prove that every continuous function is Riemann integrable. 5

(b) Show that $\,_0^1 f\,dx \quad \dfrac{2}{3}$, where f is Riemann integrable function defined as

$$f(x) \quad \begin{matrix} \dfrac{1}{2^n}\,, & \text{when } 2^{\;(n \quad 1)} \quad x \quad 2^{\;n}, \; n \quad 0, 1, 2, \cdots \\ 0\,, & \text{when } x \quad 0 \end{matrix}$$

5

UNIT—II

3. *(a)* Prove that the improper integral $\int_a^b (x-a)^{-n}\,dx$ converges if and only if $n<1$. 4

(b) Discuss the convergence of beta function. 6

4. Show that the improper integral $\int_0^\infty f\,dx$ converges conditionally, where

$$f(x)=\begin{cases} 1, & 0\le x\le 1 \\ 0, & n-1<x\le n-\dfrac{1}{n}, \quad n=2,3,4,\cdots \\ (-1)^{n-1}, & n-\dfrac{1}{n}<x\le n \end{cases}$$

10

UNIT—III

5. Let f, f_y be continuous in $[a, b; c, d]$ and let g_1, g_2 be two functions derivable in $[c, d]$ such that for all $y\in[c, d]$ the points $(g_1(y), y)$ and $(g_2(y), y)$ belong to the $[a, b; c, d]$. Then prove that $\phi(y)=\int_{g_1(y)}^{g_2(y)} f(x, y)\,dx$ is derivable in $[c, d]$ for all $y\in[c, d]$ and

$$\phi'(y)=\int_{g_1(y)}^{g_2(y)} f_y(x, y)\,dx - g_1'(y)f(g_1(y), y) + g_2'(y)f(g_2(y), y)$$

10

6. *(a)* Show that

$$\int_0^{\pi/2}\log(1-x^2\cos^2\theta)\,d\theta=\pi\log[1+\sqrt{1-x^2}]-\pi\log 2]$$

5

(b) If f is continuous in $[a, b; c, d]$, then show that

$$\int_c^d\int_a^b f(x, y)\,dx\,dy=\int_a^b\int_c^d f(x, y)\,dy\,dx$$

5

UNIT—IV

7. *(a)* Evaluate $\quad x^2y^2dxdy$ over the region $x^2 \quad y^2 \quad 1$. 5

(b) Show that $\;_C\{(x \quad y)^3dx \quad (x \quad y)^3dy\} \quad 3 \quad a^4$. 5

8. *(a)* Find the value of integral

$$_0 \quad _0 \frac{e^{\quad y}}{y}dxdy$$

by changing the order of integration. 5

(b) Evaluate

$$\frac{xy}{\sqrt{1 \quad y^2}}dxdy$$

over the positive quadrant of the circle $x^2 \quad y^2 \quad 1$. 5

UNIT—V

9. State and prove Cauchy's criterion of uniform convergence of a sequence $\{f_n\}$ of real-valued functions on a set E. 2+8=10

10. *(a)* Show that the sequence $\dfrac{n}{x \quad n}$ is uniformly convergent in $[0, k]$ whatever k may be, but not uniform in $[0, \quad)$. 5

(b) Examine term-by-term integration for the series $\quad f_n(x)$ for which

$$f_n(x) \quad n^2x(1 \quad x)^n, \; x \quad [0, 1]$$

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