MATH/VI/09 **Student's Copy**

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(6th Semester)

MATHEMATICS

Paper : Math–361

(Modern Algebra)

Full Marks : 75

Time : 3 hours

(PART : A—OBJECTIVE)

(*Marks* : 25)

The figures in the margin indicate full marks for the questions

Answer **all** questions

SECTION—A

(*Marks* : 10)

Tick (✓) the correct answer in the brackets provided : 1×10=10

1. A subgroup H of a group G is a normal subgroup of G if

(a) H is of index 1 in G ( )

(b) H is of index 2 in G ( )

(c) H is of index 3 in G ( )

(d) H is of index infinity in G ( )

2. If the order of a group G with centre Z is p^n, where p is a prime number, then

(a) $Z \quad \{e\}$ ( )

(b) $Z \quad e$ ( )

(c) $Z \quad \{e\}$ ( )

(d) Z ( )

3. A skew field

(a) is necessarily commutative ( )

(b) has no zero divisors ( )

(c) may not possess the unity element ( )

(d) has no invertible element ( )

4. The characteristic of the ring $(I_6, \ _6, \ _6)$ where $I_6 \quad \{0, 1, 2, 3, 4, 5\}$ is

(a) 0 ( )

(b) 3 ( )

(c) 6 ( )

(d) infinite ( )

5. A non-zero integer has

(a) no associates ( )

(b) exactly one associate ( )

(c) exactly two associates ( )

(d) infinite number of associates ( )

6. Let a and b be two elements of a Euclidean ring R. If b is a unit in R, then

(a) $d(ab) \quad d(a)$ ( )

(b) $d(ab) \quad d(a)$ ( )

(c) $d(ab) \quad d(a)$ ( )

(d) $d(ab) \quad d(b)$ ( )

7. Which of the following is not a subspace of R^3, where R is the set of all real numbers?

(a) $S \quad \{(x, y, z) : x \quad z \quad 0\}$ ( )

(b) $S \quad \{(x, y, z) : x \quad y \quad 2z \quad 0\}$ ( )

(c) $S \quad \{(x, y, z) : x \quad 0\}$ ( )

(d) $S \quad \{(x, y, z) : x \quad 3y \quad R\}$ ( )

8. The necessary and sufficient condition of a vector space $V(F)$ to be a direct sum of its two subspaces U and W is

(a) $V \quad U \quad W$ and $U \quad W \quad 0$ ( )

(b) $V \quad UW$ and $U \quad W \quad \{0\}$ ( )

(c) $V \quad U \quad W$ and $U \quad W \quad \{0\}$ ( )

(d) $V \quad U \quad W$ and $U \quad W \quad \{0\}$ ( )

9. If A and B are similar matrices, then

(a) $|A \quad I| \;\; |B \quad I|$ ( )

(b) $|A \quad I| \;\; |B \quad I|$ ( )

(c) $|A \quad I| \;\; |B \quad I|$ ( )

(d) $|A \quad I| \;\; |B \quad I|$ ( )

10. An $n \quad n$ matrix A over the field F is diagonalizable if and only if

(a) A has n linearly dependent eigenvectors ( )

(b) A has n linearly independent eigenvectors ( )

(c) A has n^2 linearly dependent eigenvectors ( )

(d) A has n^2 linearly independent eigenvectors ( )

SECTION—B

(*Marks* : 15)

Answer the following questions : 3×5=15

1. Prove that a subgroup H of a group G is a normal subgroup of G if and only if the product of two right cosets of H in G is a right coset of H in G.

2. Show that every ideal of a ring R is a subring of R but the converse is not necessarily true.

3. Prove that any two non-zero elements of an integral domain with unity are associates if and only if they are divisors of each other.

4. Prove that every linearly independent subset of a finitely generated vector space V is either a basis of V or can be extended to form a basis of V.

5. Prove that two eigenvectors of a square matrix A over a field F corresponding to two distinct eigenvalues of A are linearly independent.

(PART : B—DESCRIPTIVE)

(*Marks* : 50)

The figures in the margin indicate full marks for the questions

Answer **five** questions, taking **one** from each Unit

UNIT—1

1. *(a)* If H is a normal subgroup of G and K be a normal subgroup of G containing H, then prove that $\frac{G}{K} \quad \frac{G/H}{K/H}$. 7

(b) Let f be an isomorphic mapping of a group G into a group G. Show that the order of an element of G is equal to the order of its image. 3

2. State the fundamental theorem on homomorphism of groups. Prove that the set of all inner automorphisms of a group G is a normal subgroup of the group of its automorphisms and is isomorphic to the quotient group G/Z, where Z is the center of G. 2+8=10

UNIT—2

3. *(a)* Prove that a finite commutative ring without zero divisors is a field. 5

(b) Show that a commutative ring with unity is a field if it has no proper ideal. 5

4. *(a)* Prove that in a commutative ring R, an ideal S of R is prime if and only if the factor ring R / S is an integral domain. 5

(b) Prove that the ring of integers is a principal ideal domain. 5

UNIT—3

5. *(a)* Let R be a Euclidean ring and a and b be any two elements in T, not both of which are zero. Then prove that a and b have a greatest common divisor d which can be expressed in the form $d \quad a \quad b$, for some $,\quad R$. 7

(b) Show that every field is a Euclidean ring. 3

6. *(a)* Prove that an ideal S of a Euclidean ring R is maximal if and only if S is generated by some prime element of R. 6

(b) Prove that the necessary and sufficient conditions for a non-zero element a in the Euclidean ring R to be a unit is that $d(a) \quad d(1)$. 4

UNIT—4

7. *(a)* If U and V are two subspaces of a finite dimensional vector space $V(F)$, then prove that $\dim(U \quad V) \quad \dim U \quad \dim V \quad \dim(U \quad V)$. 7

(b) Show that the vectors (1, 2, 0), (0, 3, 1), (1, 0, 1) form a basis for R^3. 3

8. *(a)* Let U be a finite dimensional vector space over the field F and let $B \quad \{ \ _1, \ _2, \cdots, \ _n\}$ be an ordered basis for U. Let V be a vector space over the same field F and let $\ _1, \ _2, \cdots, \ _n$ be any n vectors in V. Then show that there exists a unique linear transformation T from U into V such that $T(\ _i) \quad \ _i, i \quad 1, 2, \cdots, n$. 7

(b) Show that if two vectors are linearly independent, then one of them is a scalar multiple of the other. 3

UNIT—5

9. *(a)* Let V and W be vector spaces over the same field F and let T be a linear transformation from V into W. If V is finite dimensional, then prove that

$$\text{rank}(T) \quad \text{nullity}(T) \quad \dim V$$ 6

(b) Let $T : R^4 \quad R^3$ be a linear transformation defined by

$$T(x, y, z, t) \quad (x \quad y \quad z \quad t, x \quad 2z \quad t, x \quad y \quad 3z \quad 3t)$$

Find the range space, null space, rank of T and nullity of T. 4

10. *(a)* Diagonalize the matrix

$$A \quad \begin{matrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{matrix}$$ 7

(b) Show that the function $T : R^2 \quad R^2$ defined by $T(x, y) \quad (1 \quad x, y)$ is a linear transformation. 3

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