

2025

( NEP—2020 )

( 5th Semester )

**MATHEMATICS (MAJOR1)****( Advanced Calculus )***Full Marks : 75**Time : 3 hours**The figures in the margin indicate full marks for the questions***( SECTION : A—OBJECTIVE )***( Marks : 10 )*Tick  the correct answer in the boxes provided :

1×10=10

1. Let  $P^*$  be a refinement of a partition  $P$ , then for a bounded function  $f$ 

(a)  $L(P^*, f) = L(P, f)$

(b)  $U(P^*, f) \leq U(P, f)$

(c)  $U(P^*, f) \geq L(P, f)$

(d)  $L(P^*, f) \leq L(P, f)$

2. If  $f(x) = x$  over  $[0, 1]$  and

$$P = \left\{ 0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, \frac{n}{n} = 1 \right\}$$

be the partition, then the value of the lower Riemann integral  $\int_0^1 f dx$  is

(a)  $\frac{1}{2}$

(b)  $\frac{1}{3}$

(c)  $-\frac{1}{2}$

(d) None of the above

3. If  $f$  and  $g$  be two positive functions such that  $f(x) \leq g(x) \forall x \in [a, b]$ , then

(a)  $\int_a^b g dx$  converges if  $\int_a^b f dx$  converges

(b)  $\int_a^b f dx$  diverges if  $\int_a^b g dx$  diverges

(c)  $\int_a^b f dx$  converges if  $\int_a^b g dx$  converges

(d) None of the above

4. The improper integral  $\int_a^\infty \frac{1}{x^n} dx$  converges if and only if

(a)  $n < 1$

(b)  $n = 1$

(c)  $n \leq 1$

(d)  $n > 1$

5. Uniformly convergent improper integral of a continuous function is

(a) not continuous

(b) may be continuous

(c) itself continuous

(d) differentiable

6. The value of the improper integral  $\int_0^\pi \frac{dx}{a + b \cos x}$  if  $a$  is positive and  $|b| < a$  is

(a)  $\frac{2\pi}{(a^2 - b^2)^{1/2}}$

(b)  $\frac{2\pi}{(a^2 - b^2)^{3/2}}$

(c)  $\frac{\pi}{(a^2 - b^2)^{3/2}}$

(d)  $\frac{\pi}{(a^2 - b^2)^{1/2}}$

7. The value of the integral  $\int_C \frac{dx}{x+y}$ , where  $C$  is the curve  $x = at^2$ ,  $y = 2at$ ,  $0 \leq t \leq 2$  is

(a)  $\frac{1}{6}$

(b)  $\log 4$

(c)  $-\frac{1}{6}$

(d)  $\log 2$

8. The value of  $\int_C 4x^3 ds$  where  $C$  is the line segment from  $(-2, -1)$  to  $(1, 2)$  is

(a)  $-15\sqrt{2}$

(b)  $15\sqrt{2}$

(c)  $-15\sqrt{3}$

(d)  $-25\sqrt{2}$

9. The sequence  $\{f_n\}$  of continuous function is uniformly convergent to a function  $f$  on  $[a, b]$ . Then  $f$  is also

(a) uniformly convergent

(b) continuous

(c) integrable

(d) differentiable

10. By  $M_n$ -test, the sequence  $\{f_n\}$  converges uniformly to  $f$  on  $[a, b]$  if and only if

(a)  $M_n \rightarrow 0$  as  $n \rightarrow 0$

(b)  $M_n \rightarrow \infty$  as  $n \rightarrow 0$

(c)  $M_n \rightarrow 0$  as  $n \rightarrow \infty$

(d) None of the above

**( SECTION : B—SHORT ANSWERS )**

( Marks : 15 )

Answer five questions, taking at least one from each Unit :

3×5=15

UNIT—I

1. Show that  $f(x) = x^2$  is  $R$ -integrable on any interval  $[0, k]$ .

2. If a function  $f$  is monotonic on  $[a, b]$ , then prove that it is  $R$ -integrable over  $[a, b]$ .

UNIT—II

3. Examine the convergence of the function

$$\int_0^{\infty} \frac{x \tan^{-1} x}{(1+x^4)^{1/3}} dx$$

4. Establish the uniform convergence of the improper integral

$$\int_0^{\infty} \frac{y}{x^2 + y^2} dx$$

with  $(0 < c \leq y \leq d)$ .

UNIT—III

5. Show that

$$\int_C [(x-y)^3 dx + (x-y)^3 dy] = 3\pi a^4$$

taken along the circle  $x^2 + y^2 = a^2$  in the counterclockwise sense.

6. Evaluate the double integral

$$\int_{\pi/2}^{\pi} \int_0^{x^2} \frac{1}{x} \cos \frac{y}{x} dy dx$$

UNIT—IV

7. Prove that the sequence  $f_n(x) = nx e^{-nx^2}$  is pointwise convergent but not uniformly on  $[0, \infty[$  by evaluating the pointwise limit.

8. Show by  $M_n$ -test, the sequence of function

$$f_n(x) = \frac{x}{1+nx^2} \quad \forall x \in \mathbb{R}$$

converges uniformly on any closed interval.

( SECTION : C—DESCRIPTIVE )

( Marks : 50 )

Answer five questions, taking at least one from each Unit :

10×5=50

UNIT—I

1. (a) Prove that a necessary and sufficient condition for the integrability of bounded function  $f$  is that to every  $\varepsilon > 0$ , there exists a corresponding  $\delta > 0$  such that  $U(P, f) - L(P, f) < \varepsilon$  for every partition  $P$  of  $[a, b]$  with norm  $\mu(P) < \delta$ .

5

- (b) Show that  $f(x) = 3x + 1$  is  $R$ -integrable on  $[1, 2]$  and prove that

$$\int_1^2 f(x) dx = \frac{11}{2}$$

5

2. (a) State Darboux's theorem and apply it to show that if  $f$  is bounded and integrable on  $[a, b]$ , then to every  $\varepsilon > 0$ ,  $\exists \delta > 0$  such that for every partition  $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$  with norm  $\mu(P) < \delta$  and for every choice of  $t_r \in [x_{r-1}, x_r]$

$$\left| \sum_{r=1}^n f(t_r)(x_r - x_{r-1}) - \int_a^b f(x) dx \right| < \varepsilon$$

1+4=5

- (b) Compute the value of  $\int_{-1}^1 f dx$ , where  $f(x) = |x|$  by dividing the interval  $[-1, 1]$  into  $2n$ -equal sub-intervals.

5

UNIT—II

3. (a) By Cauchy's test, show that  $\int_0^{\infty} \frac{\sin x}{x} dx$  is convergent.

- (b) Prove that the integral  $\int_0^{\infty} x^{n-1} e^{-x} dx$  is convergent if and only if  $n > 0$ .

4. (a) Let  $\phi(y) = \int_a^b f(x, y) dx$ , is continuous and  $f_y$  also exists and continuous in  $[a, b; c, d]$ , then prove that  $\phi$  is derivable and

$$\phi'(y) = \int_a^b f_y(x, y) dx \quad \forall y \in [c, d] \quad 5$$

- (b) Test the uniform convergence of the convergent improper integral

$$\int_0^\infty e^{-x^2} \cos yx dx$$

in  $(-\infty, \infty)$ .

5

### UNIT—III

5. (a) Evaluate the integral

$$\int_0^3 \int_{x^2}^9 x^3 e^{y^3} dy dx$$

by reversing the order of integration with a rough figure.

5

- (b) Show that

$$\int_0^1 \left\{ \int_0^1 \frac{x^2 - y^2}{x^2 + y^2} dy \right\} dx = \int_0^1 \left\{ \int_0^1 \frac{x^2 - y^2}{x^2 + y^2} dx \right\} dy$$

i.e., change in order of integration is permissible.

5

6. (a) Change the order of integration in the integral

$$\int_0^1 dx \int_0^{\sqrt{1-x^2}} \frac{e^y dy}{(1+e^y)\sqrt{1-x^2-y^2}}$$

and hence evaluate it with a rough figure.

5

- (b) Prove that

$$\int_0^1 \left\{ \int_0^1 \frac{(x-y)}{(x+y)^3} dy \right\} dx \neq \int_0^1 \left\{ \int_0^1 \frac{(x-y)}{(x+y)^3} dx \right\} dy$$

i.e., change in order of integration is not permissible.

5

7. (a) Show that  $f_n(x) = e^{-nx}$  is pointwise but not uniform convergent in  $[0, \infty[$ . Also show that the convergence is uniform in  $[k, \infty[$ , where  $k$  is any positive number. 5

(b) Show that a sequence of function  $\{f_n(x)\}$  defined on  $[a, b]$  is said to converge uniformly to  $f$  on  $[a, b]$  if and only if to each  $\varepsilon > 0$  and  $\forall x \in [a, b]$ , there exists an integer  $m$  such that

$$|f_{n+p}(x) - f_n(x)| < \varepsilon \quad \forall n \geq m, p \geq 1 \quad 5$$

8. (a) Show that the series

$$\sum_{n=1}^{\infty} \frac{x}{n^3 + n^4 x^2}$$

is uniformly convergent for all values of  $x$  and that the derivative of the sum with respect to  $x$  is given by term-by-term differentiation. 5

(b) If a sequence  $\{f_n\}$  converges uniformly to  $f$  on  $x \in [a, b]$  and let  $f_n$  be integrable  $\forall n$ , then prove that  $f$  is integrable and

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \int_a^b f_n(x) dx \quad 5$$

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