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(CBCS)

(6th Semester)

MATHEMATICS

TENTH PAPER

(Advanced Calculus)

Full Marks : 75

Time : 3 hours

(PART : A—OBJECTIVE)

(*Marks* : 25)

SECTION—A

(*Marks* : 10)

Answer **all** questions

Each question carries 1 mark

Put a Tick ☑ mark against the correct alternative in the box provided :

1. P_1 is a refinement of P_2, if and only if

(a) $P_2 \quad P_1$ ☐

(b) $P_2 \quad P_1$ ☐

(c) $P_1 \quad P_2$ ☐

(d) None of the above ☐

2. If $f(x) \quad \begin{matrix} 0, \text{ when } x & \mathbb{N} \\ 1, \text{ when } x & \mathbb{R} \quad \mathbb{N} \end{matrix}$ and $g(x) \quad \frac{1}{x}$, for all $x \quad [0, 1]$, then

(a) $f \quad R, \; g \quad R$ ☐

(b) $f \quad R, \; g \quad R$ ☐

(c) $f \quad R, \; g \quad R$ ☐

(d) $f \quad R, \; g \quad R$ ☐

3. If ${}_{a}\ f(x)\,dx$ converge conditionally, then

(a) both ${}_{a}\ f\,dx$ and ${}_{a}\ |f|\,dx$ converge ☐

(b) both ${}_{a}\ f\,dx$ and ${}_{a}\ |f|\,dx$ diverge ☐

(c) ${}_{a}\ f\,dx$ diverges and ${}_{a}\ |f|\,dx$ converges ☐

(d) ${}_{a}\ f\,dx$ converges and ${}_{a}\ |f|\,dx$ diverges ☐

4. The integral ${}_{0}\ e^{n\ 1}e^{\ x}\,dx$ is

(a) convergent when $n\ \ 1$ and divergent when $n\ \ 1$ ☐

(b) convergent when $n\ \ 1$ and divergent when $n\ \ 1$ ☐

(c) convergent when $n\ \ 0$ and divergent when $n\ \ 0$ ☐

(d) convergent when $n\ \ 0$ and divergent when $n\ \ 0$ ☐

5. If the function $f(x, y)$ and $f_n(x, y)$ exist and continuous in $[a, b; c, d]$, then

(a) derivative of ${}_{a}^{b} f(x, y)\,dx$ with respect to y is not possible to determine ☐

(b) derivative of ${}_{a}^{b} f(x, y)\,dx$ with respect to y is always possible to determine ☐

(c) derivative of ${}_{a}^{b} f(x, y)\,dx$ with respect to y is not always continuous ☐

(d) None of the above ☐

6. Consider the following Assertion (A) and Reason (R) :

Assertion (A) : If $g(y) \quad y$ and $h(y) \quad 1$, then $\frac{d}{dy} {}_{h(y)}^{g(y)} xy\, dx \quad \frac{3y^2 \quad 1}{2}$.

Reason (R) : First integrate with respect to x and then differentiate with respect to y gives the result.

(a) (A) is right and (R) is wrong ☐

(b) Both (A) and (R) are wrong ☐

(c) Both (A) and (R) are right ☐

(d) (A) is wrong and (R) is right ☐

7. Choose the correct one.

(a) $_C f\, dx \quad g\, dy \qquad _C f\, dx \quad g\, dy$ ☐

(b) $_C f\, dx \quad g\, dy \qquad _C f\, dx \quad g\, dy$ ☐

(c) $_C f\, dx \quad g\, dy \qquad _C f\, dx \quad g\, dy$ ☐

(d) $_C f\, dx \quad g\, dy \qquad _C f\, dx \quad g\, dy$ ☐

8. Which of the following is a Jordan curve?

(a) Parabola ☐ *(b)* Hyperbola ☐

(c) Straight line ☐ *(d)* Ellipse ☐

9. Which of the following is correct?

(a) The sequence $\{x^n\}$ is convergent uniformly on $[0, 1]$ ☐

(b) The sequence $\{x^n\}$ is convergent uniformly on $[0, \quad)$ ☐

(c) The sequence $\{x^n\}$ is convergent uniformly on $[0, k]$, $0 \quad k \quad 1$ ☐

(d) None of the above ☐

10. Choose the odd one.

(a) Harmonic series is divergent ☐

(b) Absolutely convergent series is convergent in $\mathbb{R}$ ☐

(c) Both *(a)* and *(b)* are correct ☐

(d) Neither *(a)* nor *(b)* is correct ☐

SECTION—B

(*Marks* : 15)

Each question carries 3 marks

1. Explain briefly the Riemann integrability of the function

$$f(x) \quad \begin{cases} 0, \text{ when } x \quad (\frac{1}{2}, 1] \\ \frac{1}{2}, \text{ when } x \quad (\frac{1}{4}, \frac{1}{2}] \\ 1 \text{ , otherwise} \end{cases}$$

OR

Show that a constant function is Riemann integrable.

2. Show that $_1 \; \frac{\sin x}{x^4} dx$ is absolutely convergent.

OR

Prove that $_0 \; \frac{e^{\;ax} \quad e^{\;bx}}{x} dx \quad \log\frac{b}{a}$, where a and b are constants.

3. Test the uniform convergence of improper integral $_0 \; e^{\;x^2} \sin(xy)\, dx$ in the interval (,).

OR

Write the statement of Weierstrass M-test for uniform convergence of improper integral of $_a^b f(x, y)\, dx$ type.

4. Calculate $_A \; \frac{\sin x}{x} dA$, where A is the region in the XY-plane bounded by $y \quad x$ and $x \quad$.

OR

Prove that $_0 \; _0^{\sin x} y \, dy dx \quad \frac{}{2}$.

5. Write the necessary and sufficient condition of uniform convergence for sequential series.

OR

Show that the sequence $\{\sin(nx \quad n) / n\}$ for any real number x and natural number n is convergent to zero.

(PART : B—DESCRIPTIVE)

(*Marks* : 50)

The figures in the margin indicate full marks for the questions

Answer **one** question from each Unit

UNIT—I

1. *(a)* If $f : [0, 1] \quad [0, 1]$ defined by

$$f(x) \quad \frac{2^k \quad 1}{2^k} \quad x \quad \frac{2^{k \quad 1} \quad 1}{2^{k \quad 1}}, \frac{2^k \quad 1}{2^k}, \; k \quad 1$$

prove that f is Riemann integrable and $\;_0^1 f(x)\, dx \quad \frac{2}{3}$. 6

(b) A bounded function f is defined as

$$f(x) \quad \begin{matrix} \frac{p}{q}, \text{ if } x \text{ is any non-zero rational } \frac{p}{q} \text{ in its lowest term} \\ 0 \text{ , if } x \text{ is irrational or zero} \end{matrix}$$

Show that f is R-integrable in [0, 1] and the value of the integral is zero. 4

2. *(a)* Let $g \quad [a, b]$ and let f be monotonic and non-negative on $[a, b]$. Then for some and of $[a, b]$, prove that

$$\;_a^b f(x)\, g(x)\, dx \quad f(a) \;_a \; g\, dx \;\; or \;\; \;_a^b f(x)\, g(x)\, dx \quad f(a) \;^b \; g(x)\, dx$$ 6

(b) Show that the function f defined as

$$f(x) \quad \begin{matrix} \frac{1}{2^n} \text{ , when } \frac{1}{2^{n \quad 1}} \quad x \quad \frac{1}{2^n} \\ 0 \quad \text{ , when } x \quad 0 \end{matrix}$$

is R-integrable on [0, 1], although it has an infinite number of points of discontinuity. 4

UNIT—II

3. *(a)* Evaluate $\;_1 \frac{\cos ax \quad \cos bx}{x} dx$. 3

(b) Discuss the convergence of $\;_1 x^{n \quad 1} e^{\quad x} dx$. 7

4. *(a)* If is bounded and monotonic in $[a, \ )$ and $\ _{a} f\, dx$ convergent at ,

then prove that $\ _{a} f \ \ dx$ is convergent at . 6

(b) Show that the integral $\ _{0} \dfrac{\sin x}{x} dx$ converges. 4

UNIT—III

5. *(a)* If $a \ \ b$, then show that

$$\ _{0}^{\overline{2}} \log \frac{a \quad b \sin}{1 \quad b \sin} \frac{d}{\sin} \qquad \sin^{\ 1} \frac{b}{a}$$ 4

(b) If $a \ \ 0$, $|b| \ \ a$, evaluate

$$\ _{0} \frac{dx}{a \quad b \cos x}$$

and deduce that $\ _{0} \dfrac{dx}{(a \quad b \cos x)^2} \qquad \dfrac{a}{(a^2 \quad b^2)^{3/2}}$.

Also find $\ _{0} \dfrac{\cos x\, dx}{(a \quad b \cos x)^3}$. 6

6. Let f, f_y be continuous in $[a, b; c, d]$ and let g_1, g_2 be two functions derivable in $[c, d]$ such that for all $y \ \ [c, d]$ the points $(g_1(y), y)$ and $(g_2(y), y)$ belong to the $[a, b; c, d]$. Then prove that

$$(y) \quad \ _{g_1(y)}^{g_2(y)} f(x, y)\, dx \text{ is derivable in } [c, d] \text{ for all } y \quad [c, d]$$

and $$(y) \quad \ _{g_1(y)}^{g_2(y)} f_y(x, y)\, dx \quad g_1(y) f(g_1(y), y) \quad g_2(y) f(g_2(y), y)$$ 10

UNIT—IV

7. *(a)* Discuss the integrability of bounded function over rectangle. 5

(b) Evaluate $\ _{R}(x \quad y)\, dxdy$ over the rectangle $R \ \ [a, b; c, d]$. 5

8. *(a)* Prove that $\int_0^1 \int_0^1 \frac{x \quad y}{(x \quad y)^3}\, dydx \quad \frac{1}{2}$. Also prove that the value becomes $\quad \frac{1}{2}$, if we interchange dx and dy. 6

(b) Change the order of integration $\int_0^{2\sqrt{\ }} \int_0^{\sqrt{\ }} \sin x^2\, dydx$ and evaluate it. 4

UNIT—V

9. *(a)* Show that the sequence $\{f_n\}$, $f_n(x) \quad nxe^{\quad nx^2}$ is pointwise but not uniformly convergent in $[0, \quad)$. 7

(b) Show that the series, $\cos x \quad \frac{\cos 2x}{2^2} \quad \frac{\cos 3x}{3^2} \quad \cdots$, converges uniformly on $\mathbb{R}$. 3

10. State and prove Cauchy's criterion of uniform convergence of a sequence $\{f_n\}$ of a real-valued function on a set E. 2+8=10

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