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(CBCS)

(6th Semester)

MATHEMATICS

NINTH PAPER

(Modern Algebra)

Full Marks : 75

Time : 3 hours

(PART : A—OBJECTIVE)

(*Marks* : 25)

Answer **all** questions

SECTION—A

(*Marks* : 10)

Each question carries 1 mark

Put a Tick ☑ mark against the correct answer in the box provided :

1. If G is a non-Abelian group of order p^3, where p is prime, then the order of the center Z of G is

(a) 1 ☐ *(b)* p ☐

(c) p^2 ☐ *(d)* p^3 ☐

2. A subgroup H of a group G is normal, if it is of index

(a) 0 ☐ *(b)* 1 ☐

(c) 2 ☐ *(d)* 3 ☐

3. Which of the following is ***not*** an integral domain?

(a) The ring of integers ☐

(b) The ring of all 2 2 matrices with elements as integers ☐

(c) $(\{0, 1, 2, 3, 4\}, \ _5, \ _5)$ ☐

(d) The ring of all real numbers ☐

4. The ring of Gaussian integers is not

(a) commutative with respect to addition ☐

(b) commutative with respect to multiplication ☐

(c) a field ☐

(d) an integral domain ☐

5. Let a and b be two non-zero elements in a Euclidean ring R, then b is a non-unit in R, if

(a) $d(ab) \quad d(a)$ ☐

(b) $d(ab) \quad d(a)$ ☐

(c) $d(ab) \quad d(a)$ ☐

(d) $d(ab) \quad d(ba)$ ☐

6. In the quadratic ring of integers $Z[i\sqrt{5}] \quad \{a \quad i\sqrt{5}b;\ a,\ b \quad Z\}$, the number 3 is

(a) irreducible but not prime ☐

(b) prime but not irreducible ☐

(c) irreducible and prime ☐

(d) neither irreducible nor prime ☐

7. Which of the following set of vectors is linearly dependent?

(a) $\{(2, 1, 4), (1, \quad 1, 2), (3, 1, \quad 2)\}$ ☐

(b) $\{(\quad 1, 2, 1), (3, 0, \quad 1), (\quad 5, 4, 3)\}$ ☐

(c) $\{(1, 2, 0), (0, 3, 1), (\quad 1, 0, 1)\}$ ☐

(d) $\{(2, \quad 3, 1), (3, \quad 1, 5), (1, \quad 4, 3)\}$ ☐

8. If $V(F)$ is a vector space with zero element 0 and if U and W are disjoint subspaces of $V(F)$, then

(a) $U \quad V$ ☐

(b) $U \quad V \quad 0$ ☐

(c) $U \quad V \quad \{0\}$ ☐

(d) $U \quad V \quad 0$ ☐

9. Let $T : R^3 \quad R^3$ be a linear transformation whose nullity is 2. Then the rank of T is

(a) 0 ☐ *(b)* 1 ☐

(c) 2 ☐ *(d)* 3 ☐

10. Which of the following functions is a linear transformation from R^2 into R^2?

(a) $f(x, y) \quad (x^2, y), \quad (x, y) \quad R^2$ ☐

(b) $f(x, y) \quad (\sin x, y), \quad (x, y) \quad R^2$ ☐

(c) $f(x, y) \quad (1 \quad x, y), \quad (x, y) \quad R^2$ ☐

(d) $f(x, y) \quad (x \quad y, 0), \quad (x, y) \quad R^2$ ☐

SECTION—B

(*Marks* : 15)

Each question carries 3 marks

Answer the following questions :

1. *(a)* If the order of a group G is p^2, where p is a prime number, then prove that G is Abelian.

OR

(b) Show that every quotient group of an Abelian group is Abelian and the converse is not necessarily true.

2. *(a)* Prove that every field is an integral domain.

OR

(b) Show that the set of all rational numbers is a subring but not an ideal of the ring of all real numbers.

3. *(a)* Let f be a homomorphism of a ring R into a ring R. Show that the kernel of f is an ideal of R.

OR

(b) Prove that the necessary and sufficient condition for a non-zero element a in a Euclidean ring to be a unit is that $d(a) \quad d(1)$.

4. *(a)* Show that the set $W \quad \{(x, y, z) \quad x \quad 3y \quad 4z \quad 0\}$ is a subspace of R^3.

OR

(b) Show that the vectors $\{(2, \quad 3, 1), (3, \quad 1, 5), (1, \quad 4, 3)\}$ forms a basis for R^3.

5. *(a)* Let $T: R^2 \quad R^2$ be a linear operator defined by $T(x, y) \quad (4x \quad 2y, 2x \quad y)$. Determine the matrix T with respect to the ordered basis $\{(1, 1), (\quad 1, 0)\}$.

OR

(b) Let $T: R^3 \quad R^2$ be a linear transformation defined by $T(x, y, z) \quad (x \quad y, y \quad z)$. Find the kernel of T.

(PART : B—DESCRIPTIVE)

(*Marks* : 50)

The figures in the margin indicate full marks for the questions

Answer **five** questions, taking **one** from each Unit

UNIT—I

1. *(a)* State the fundamental theorem on homomorphism of groups. Hence if G is a group and H is any subgroup of G and if N is any normal subgroup of G, then prove that

$$\frac{HN}{N} \quad \frac{H}{H \quad N}$$

1+6=7

(b) Prove that a subgroup H of a group G is a normal subgroup of G if and only if each left coset of H in G is a right coset of H in G. 3

2. *(a)* Prove that the set $I(G)$ of all inner automoprhisms of a group G is a normal subgroup of the group $A(G)$ of all automorphisms of G. Also prove that $I(G)$ is isomorphic to the quotient group G / Z of G, where Z is the center of G. 7

(b) Show that for an Abelian group, the only inner automorphism is the identity mapping but for non-Abelian groups, there exist non-trivial automorphisms. 3

UNIT—II

3. *(a)* Prove that a ring R is without zero divisors, if and only if the cancellation laws hold in R. 4

(b) Prove that the ring of integers is a principal ideal domain. 6

4. *(a)* Prove that an ideal S of a commutative ring R with unity is maximal, if and only if the residue class R / S is a field. 6

(b) Prove that a commutative ring with unity is a field if it has no proper ideal. 4

UNIT—III

5. *(a)* Let a and b be any two elements of a Euclidean ring R, not both of which are zero. Prove that a and b have a greatest common divisor d which can be expressed in the form

$$d \quad a \quad b \text{ for some } \quad , \quad R$$ 6

(b) Let D be an integral domain with unity element 1. Show that two non-zero elements a, b $\;$ D are associates, if and only if a / b and b / a. 4

6. *(a)* Let R be a Euclidean domain and a be a non-zero non-unit element in R. If $a \quad p_1 p_2 \cdots p_3 \quad q_1 q_2 \cdots q_n$, where p s and q s are prime elements of R, then show that $m \quad n$ and each p is an associate of some q and each q is an associate of some p. 6

(b) Define a unique factorization domain. Show that the greatest common divisor of two elements of a unique factorization domain is unique up to associates. 4

UNIT—IV

7. *(a)* Prove that every linearly independent subset of a finitely generated vector space $V(F)$ is either a basis of V or can be extended to form a basis of V. 6

(b) Prove that the union of two subspaces is a subspace if and only if one is contained in the other. 4

8. *(a)* If U and W are two subspaces of a finite dimensional vector space $V(F)$, then prove that $\dim(U \quad W) \quad \dim U \quad \dim W \quad \dim(U \quad W)$. 7

(b) If $V(F)$ is a finitely generated vector space of dimension n, then show that any set of n linearly independent vectors in V forms a basis of V. 3

UNIT—V

9. *(a)* Let U be a finite dimensional vector space over the field F and let $B \quad \{ \ _1, \quad _2, \cdots \quad _n\}$ be an ordered basis for U. Let V be a vector space over the same field F and let $\ _1, \quad _2, \cdots, \quad _n$ be any n vectors in V. Then prove that there exists a unique linear transformation T from U into V such that $T(\ _i) \quad \ _i$, $i \quad 1$ to n. 6

(b) Let $T : R^3 \quad R^3$ be a linear transformation defined by

$$T(x, y, z) \quad (x \quad 2y \quad z, y \quad z, x \quad y \quad 2z)$$

Find the range space, null space, rank of T and nullity of T. 4

10. *(a)* Let V and W be vector spaces over the same field F and let T be a linear transformation from V into W. If V is finite dimensional, then prove that rank T + nullity T = dim V. 5

(b) Let U be a finite dimensional vector space of dimension n over the field F and V be a finite dimensional vector space of dimension m over the same field F. Then prove that the vector space $L(U, V)$ of all linear transformations from U into V is also finite dimensional and is of dimension mn. 5

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