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(CBCS)

(5th Semester)

MATHEMATICS

SEVENTH PAPER (Math–353)

(Complex Analysis)

Full Marks : 75

Time : 3 hours

(PART : A—OBJECTIVE)

(*Marks* : 25)

Answer **all** questions

SECTION—A

(*Marks* : 10)

Each question carries 1 mark

Put a Tick ☑ mark against the correct answer in the box provided :

1. If z_1 i and z_2 $\sqrt{3}$ i, then the arg $\frac{z_1}{z_2}$ is

(a) $\frac{}{3}$ ☐

(b) $\frac{5}{6}$ ☐

(c) $\frac{4}{3}$ ☐

(d) $\frac{3}{4}$ ☐

2. The value of z^4 for z 1 i is

(a) $2i$ ☐ (b) 4 ☐

(c) $3\sqrt{3}$ i ☐ (d) $\sqrt{3}$ i ☐

3. The value of z for which the function w defined by z e $^{v}(\cos u$ $i \sin u)$, where w u iv ceases to be analytic is

(a) i ☐ (b) i ☐

(c) 0 ☐ (d) ☐

4. The function $f(z)$ $|xy|^{1/2}$

(a) is everywhere continuous and analytic ☐

(b) is everywhere continuous but not analytic ☐

(c) is discontinuous but analytic everywhere ☐

(d) Cauchy-Riemann equation is satisfied at the origin but not analytic at that point ☐

5. A sequence with more than one limit point

(a) is convergent always ☐

(b) may be convergent ☐

(c) cannot convergent ☐

(d) None of the above ☐

6. A power series within its circle of convergence

(a) converges absolutely ☐

(b) converges uniformly ☐

(c) converges absolutely and uniformly ☐

(d) is divergent ☐

7. For an arc L, the value of $\;_{L}|dz|$ is equal to

(a) the length of the arc ☐

(b) $2\ \ i$ ☐

(c) 0 ☐

(d) None of the above ☐

8. The integral of $\bar{z}$ along the above part of real axis of a semi-circular arc $|z|\quad 1$ is

(a) $\ \ i$ ☐

(b) $2\ \ i$ ☐

(c) $\ \ i$ ☐

(d) $\ 2\ \ i$ ☐

9. If a function $f(z)$ is analytic within and on a circle C, then Taylor's series is valid for

(a) every point inside C ☐

(b) every point on the circle C ☐

(c) some annulus ☐

(d) every point inside and on a circle C ☐

10. The nature of the point $z\quad 0$, for the function $f(z)\quad e^{\frac{1}{z}}$ is

(a) removable singularity ☐

(b) pole ☐

(c) essential singularity ☐

(d) zero ☐

SECTION—B

(*Marks* : 15)

Each question carries 3 marks

1. Find an upper bound for $\left| \dfrac{1}{z^4 \quad 5z \quad 1} \right|$, if $|z| \quad 2$.

OR

Find the condition for which the four points z_1, z_2, z_3 and z_4 to be concyclic.

2. For what value of z the function w defined as $z \quad \sinh u \cos v \quad i \cosh u \sin v$, where $w \quad u \quad iv$ ceases to be analytic?

OR

If $f(z)$ is a regular function of z, then prove that

$$\frac{2}{x^2} \quad \frac{2}{y^2} \log |f(z)| \quad 0$$

3. For what value of z the power series $\;_{n \quad 1} \dfrac{1}{(z^2 \quad 1)^n}$ will converge?

OR

Find the radius of convergence of the series

$$\frac{z}{2} \quad \frac{1.3}{2.5} z^2 \quad \frac{1.3.5}{2.5.6} z^3 \quad \cdots$$

4. Evaluate $\;_L \dfrac{z \quad 2}{z} dz$, where L is semi-circle $z \quad 2e^{it}$, $0 \quad t \quad$.

OR

Evaluate the integral $\;_0^{1 \quad i} z^2 dz$.

5. What kind of singularity has the function $(\sin z \quad \cos z)$ at $z \quad$?

OR

Prove the minimum modulus principle with the help of maximum modulus principle.

(PART : B—DESCRIPTIVE)

(*Marks* : 50)

The figures in the margin indicate full marks for the questions

Answer **one** question from each Unit

UNIT—I

1. (a) Prove that the sum and product of two complex numbers are real if and only if they are conjugate to each other. 4

(b) Find the loci of the point z satisfying the following conditions : 6

(i) $\left|\frac{z \quad i}{z \quad i}\right| \quad 2$

(ii) $\arg \frac{z \quad 1}{z \quad 1} \quad \overline{3}$

2. (a) Find the inverse of the point (i) with respect to the circle passing through the points 1, i, 1 i. 6

(b) Prove that $|z_1 \quad z_2| \quad |z_1| \quad |z_2|$, and deduce that

$$|z_1 \quad z_2|^2 \quad |z_1 \quad z_2|^2 \quad 2(|z_1|^2 \quad |z_2|^2)$$

where z_1 and z_2 are any complex numbers. 4

UNIT—II

3. (a) State and prove Cauchy-Riemann equation. 5

(b) Prove that $w \quad |z|^2$ is continuous everywhere but nowhere differentiable except at origin. 5

4. (a) Show that the function $u \quad \cos x \cosh y$ is harmonic. Find its harmonic conjugate and corresponding analytic function. 3

(b) If $u \quad v \quad (x \quad y)(x^2 \quad 4xy \quad y^2)$, then find the analytic function $f(z) \quad u \quad iv$ in terms of z. 3

(c) If $f(z)$ is a regular function of z, then prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|Rf(z)|^2 = 2|f'(z)|^2$$

where $Rf(z)$ represents the real part of $f(z)$. 4

UNIT—III

5. (a) State and prove Cauchy-Hadamard theorem. 6

(b) Find the radii of convergence of the following series : 4

(i) $\sum_{n=1}^{\infty}\frac{n!}{n^n}z^n$

(ii) $\sum_{n=1}^{\infty}\frac{in+2}{2^n}z^n$

6. (a) Find the circle of convergence of the following series : 6

(i) $\sum_{n=1}^{\infty}\left(\frac{z^2+1}{1+i}\right)^n n^2$

(ii) $\sum_{n=1}^{\infty}\left(\frac{iz+1}{2+i}\right)^n$

(b) Examine the behaviour of the power series $\sum_{n=1}^{\infty}\frac{z^{4n}}{4n+1}$ on the circle of convergence. 4

UNIT—IV

7. (a) A function $f(z)$ is continuous on a contour C of length l and if M be the upper bound of $|f(z)|$ on C, then show that

$$\left|\int_C f(z)\,dz\right| \le Ml$$

5

(b) If a function $f(z)$ is analytic within and on a closed contour C, then show that the derivatives of all orders are analytic and are given by

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$ 5

8. *(a)* Evaluate $\int_C \frac{e^{az}}{z^{n+1}}\,dz$, where C is a closed contour containing the origin inside it. 4

(b) Evaluate the integral $\int_C \frac{z\,dz}{(9-z^2)(z+i)}$, where C is the circle $|z| = 2$ described in the positive sense. 6

UNIT—V

9. *(a)* Let $f(z)$ is analytic in the closed ring bounded by two concentric circles C and C' of centre a and radii R and R' $(R' < R)$. If z be any point of the annulus, then show that

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} b_n(z-a)^{-n}$$

where

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(t)}{(t-a)^{n+1}}\,dt \text{ and } b_n = \frac{1}{2\pi i}\int_{C'} \frac{f(t)}{(t-a)^{-n+1}}\,dt$$ 6

(b) If $f(z)$ is an integral function satisfying the inequality $|f(z)| \le M$ for all values of z, where M is a positive constant, then prove that $f(z)$ is constant. 4

10. *(a)* Expand the function $\frac{1}{z^2-3z+2}$ in the region *(i)* $0 < |z| < 1$ and *(ii)* $1 < |z| < 2$. 5

(b) Find the kind of singularities of the following functions : 5

(i) $\sin\frac{1}{1-z}$ at $z = 1$

(ii) $z\operatorname{cosec} z$ at $z = \infty$

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