MATH/V/CC/06 **Student's Copy**

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(CBCS)

(5th Semester)

MATHEMATICS

SIXTH PAPER (Math–352)

(Real Analysis)

Full Marks : 75

Time : 3 hours

(PART : A—OBJECTIVE)

(*Marks* : 25)

SECTION—A

(*Marks* : 10)

Each question carries 1 mark

Tick (✓) the correct answer in the brackets provided :

1. Which of the following statements is False?

(a) The intersection of an arbitrary family of open sets is open ( )

(b) The union of an arbitrary family of open sets is open ( )

(c) The intersection of an arbitrary family of closed sets is closed ( )

(d) The union of a finite number of closed sets is closed ( )

2. Which of the following sets is an open set?

(a) The set of rational numbers ( )

(b) $\frac{1}{n} : n \quad N$ ( )

(c) The open interval (a, b) ( )

(d) The closed interval $[a, b]$ ( )

3. The derived set of every subset of a discrete metric space is

(a) the empty set ( )

(b) the set of all real numbers ( )

(c) the Euclidean space ( )

(d) an infinite set ( )

4. Which of the following metric spaces is not complete?

(a) The set of all real numbers with the usual metric ( )

(b) The set of all rational numbers with the usual metric ( )

(c) The discrete space ( )

(d) The space of all bounded continuous real-valued functions defined on the closed interval [0, 1] with the metric d given by $d(f, g) = \max_{0 \le x \le 1} |f(x) - g(x)|$ ( )

5. Which of the following functions is continuous at (0, 0)?

(a) $f(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$ ( )

(b) $f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$ ( )

(c) $f(x, y) = \begin{cases} \dfrac{x^2 y^2}{x^2 + y^4}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$ ( )

(d) $f(x, y) = \begin{cases} \dfrac{x^4 - y^4}{x^4 + y^4}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$ ( )

6. The image of a compact set under a continuous function is

(a) closed but not bounded ( )

(b) not closed but bounded ( )

(c) neither closed nor bounded ( )

(d) both closed and bounded ( )

7. Let f be a real-valued function with an open domain $D \quad R^n$. Then the function admits of directional derivatives at every point where

(a) it is continuous ( )

(b) it possesses continuous first-order partial derivatives ( )

(c) it possesses second-order partial derivatives ( )

(d) it possesses nth order partial derivatives ( )

8. If $y_1, y_2, \cdots, y_n$ are determined as functions of $x_1, x_2, \cdots, x_n$ by the functional equations $f_i(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n) \quad 0, i \quad 1$ to n, then

$$\frac{(f_1, f_2, \cdots, f_n)}{(x_1, x_2, \cdots, x_n)}$$

(a) $\dfrac{(f_1, f_2, \cdots, f_n)}{(y_1, y_2, \cdots, y_n)} \dfrac{(y_1, y_2, \cdots, y_n)}{(x_1, x_2, \cdots, x_n)}$ ( )

(b) $\dfrac{(f_1, f_2, \cdots, f_n)}{(y_1, y_2, \cdots, y_n)} \dfrac{(y_1, y_2, \cdots, y_n)}{(x_1, x_2, \cdots, x_n)}$ ( )

(c) $(\quad 1)^n \dfrac{(f_1, f_2, \cdots, f_n)}{(y_1, y_2, \cdots, y_n)} \dfrac{(y_1, y_2, \cdots, y_n)}{(x_1, x_2, \cdots, x_n)}$ ( )

(d) $(\quad 1)^{n \quad 1} \dfrac{(f_1, f_2, \cdots, f_n)}{(y_1, y_2, \cdots, y_n)} \dfrac{(y_1, y_2, \cdots, y_n)}{(x_1, x_2, \cdots, x_n)}$ ( )

9. The second term in the expansion of $e^{ax} \cos(by)$ is

(a) 1 ( )

(b) $ax \quad by$ ( )

(c) ax ( )

(d) by ( )

10. A necessary condition for $f(x, y)$ to have an extreme value at (a, b) is that

(a) $f_x(a, b) \quad 0 \quad f_y(a, b)$ ( )

(b) $f_{xx}(a, b) \quad 0 \quad f_{yy}(a, b)$ ( )

(c) $f_{xy}(a, b) \quad 0 \quad f_{yx}(a, b)$ ( )

(d) $f_x(a, b) \quad f_y(0, 0) \quad 0$ ( )

Section—B

(*Marks* : 15)

Each question carries 3 marks

Answer the following questions :

1. Prove that the interior of a set S is the largest open subset of S.

OR

Prove that the derived set of a set is closed.

2. Show that an infinite set with the discrete metric is not compact.

OR

Show that the discrete space is a complete metric space.

3. Prove that a function $f : D \to R$, $D \subseteq R^n$ is continuous if and only if $f^{-1}(U)$ is open in R for every open set U in R.

OR

Show that the function

$$f(x, y) = \begin{cases} \dfrac{2x^4 + 3y^4}{x^2 + y^2} & , (x, y) \neq (0, 0) \\ 0 & , (x, y) = (0, 0) \end{cases}$$

is continuous at (0, 0).

4. Let $f : D \to R$, $D \subseteq R^2$ and $(a, b) \in D$. Show that if one of the partial derivatives exists and is bounded in a neighbourhood of (a, b) and the other exist at (a, b), then the f is continuous at (a, b).

OR

Discuss the differentiability of the function

$$f(x, y) = \begin{cases} x^2 \sin\dfrac{1}{x} + y^2 \sin\dfrac{1}{y} & , \text{ if } xy \neq 0 \\ x^2 \sin\dfrac{1}{x} & , \text{ if } x \neq 0, y = 0 \\ y^2 \sin\dfrac{1}{y} & , \text{ if } x = 0, y \neq 0 \end{cases}$$

at (0, 0).

5. Prove that the function $f(x, y) \quad x^2 \quad 2xy \quad y^2 \quad x^4 \quad y^4$ has a minima at the origin.

OR

Using Taylor's theorem, show that the expansion of $\sin(xy)$ in powers of $(x \quad 1)$ and $y \quad \frac{\;}{2}$ up to and including the second-degree term is

$$1 \quad \frac{\;}{8}(x \quad 1)^2 \quad \frac{\;}{2}(x \quad 1)\; y \quad \frac{\;}{2} \quad \frac{1}{2}\; y \quad \frac{\;}{2}^{\;2}$$

(PART : B—DESCRIPTIVE)

(*Marks* : 50)

The figures in the margin indicate full marks for the questions

Answer **five** questions, selecting **one** from each Unit

UNIT—I

1. *(a)* State and prove Bolzano-Weierstrass theorem. 6

(b) If a sequence of closed intervals $[a_n, b_n]$ is such that each member $[a_{n \quad 1}, b_{n \quad 1}] \quad [a_n, b_n]$ and $\lim\limits_{n} (a_n \quad b_n) \quad 0$, then prove that there is one and only one point common to all the intervals. 4

2. *(a)* State and prove Heine-Borel theorem. 7

(b) If S and T are two subsets of the set of real numbers, show that $(S \quad T) \quad S \quad T$, where $'$ denotes derived set. 3

UNIT—II

3. *(a)* Prove that the space R^n of all ordered n-tuples with the metric d, where

$$d(x, y) \quad \sum_{i \quad 1}^{n} (x_i \quad y_i)^2 \;^{\frac{1}{2}}$$

is a complete metric space. 6

(b) Show that every closed sphere in a metric space is closed. 4

4. *(a)* Let (X, d) be any metric space. Prove that a subset of X is closed if and only if its complement is open. 5

(b) Prove that every compact subset of a metric space is closed. 5

UNIT—III

5. *(a)* Define uniform continuity of a function in R^n. Prove that a function continuous on a compact set is uniformly continuous. 1+5=6

(b) Show that the function

$$f(x, y) = \begin{cases} \dfrac{x^2 - xy + x - y}{x - y}, & (x, y) \neq (2, 2) \\ 4, & (x, y) = (2, 2) \end{cases}$$

is discontinuous at (2, 2). Mention the type of discontinuity at (2, 2). 1+3=4

6. *(a)* Prove that a set is compact if and only if every infinite subset thereof has a limit point in the set. 5

(b) Let $f : D \to R$, $D \subseteq R^n$ where D is a convex set. Show that f assumes every value between $f(x)$ and $f(y)$, $\forall x, y \in D$. 5

UNIT—IV

7. *(a)* If

$$u = \frac{x}{(1 - r^2)^{\frac{1}{2}}}, \; v = \frac{y}{(1 - r^2)^{\frac{1}{2}}}, \; w = \frac{z}{(1 - r^2)^{\frac{1}{2}}}$$

where $r = \sqrt{x^2 + y^2 + z^2}$, show that $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = \dfrac{1}{(1 - r^2)^{\frac{5}{2}}}$. 5

(b) If f_x exists throughout a neighbourhood of a point of (a, b) and $f_y(a, b)$ exists, then prove that for any point $(a + h, b + k)$ of this neighbourhood

$$f(a + h, b + k) - f(a, b) = hf_x(a + \theta h, b + k) + k\{f_y(a, b) + \eta\}$$

where $0 < \theta < 1$ and η is a function of k which tends to 0 with k. 5

8. *(a)* Prove that a function which is differentiable at a point possesses the first-order partial derivatives at that point but the converse is not necessarily true. 6

(b) Show that the function

$$f\,(x,\ y) \quad \begin{matrix} (x \quad y)\sin \dfrac{1}{x \quad y} & ,\ x \quad y \quad 0 \\ 0 & ,\ x \quad y \quad 0 \end{matrix}$$

is continuous at (0, 0) but its first-order partial derivatives do not exist at (0, 0). 4

UNIT—V

9. *(a)* State and prove Young's theorem. 1+5=6

(b) Show that the conditions of Schwarz's theorem are not satisfied for the function

$$f\,(x,\ y) \quad \begin{matrix} xy\ \dfrac{x^2 \quad y^2}{x^2 \quad y^2} & ,\ \text{when}\ x^2 \quad y^2 \quad 0 \\ 0 & ,\ \text{when}\ x \quad y \quad 0 \end{matrix}$$

at (0, 0) and $f_{xy}(0,\ 0) \quad f_{yx}(0,\ 0)$. 4

10. *(a)* State Taylor's theorem for two variables. Hence, show that for 0 1,

$$\sin x \sin y \quad xy \quad \frac{1}{6}\{(x^3 \quad 3xy^2)\cos \quad x \sin \quad y \quad (y^3 \quad 3x^2 y)\sin \quad x \cos \quad y\}.$$

1+3=4

(b) Examine the following functions for extreme values : 3+3=6

(i) $f\,(x,\ y) \quad y^2 \quad 4xy \quad 3x^2 \quad x^3$

(ii) $f\,(x,\ y) \quad x^3 \quad y^3 \quad 3x^2 \quad 3y^2 \quad 9x$

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