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(CBCS)

(5th Semester)

MATHEMATICS

SIXTH PAPER (Math–352)

(Real Analysis)

Full Marks : 75

Time : 3 hours

(PART : A—OBJECTIVE)

(*Marks* : 25)

SECTION—A

(*Marks* : 10)

Each question carries 1 mark

Tick (✓) the correct answer in the brackets provided :

1. Which of the following sets is not a closed set?

(a) The set of all real numbers ( )

(b) The empty set ( )

(c) The derived set of a set ( )

(d) The set of all rational numbers ( )

2. Every open cover of a set in R^n has

(a) a finite sub-cover ( )

(b) an infinite sub-cover ( )

(c) a countable sub-cover ( )

(d) no sub-cover ( )

3. The subset $S \quad \{(x, y) : x^2 \quad y^2 \quad 1,\ x,\ y \quad R\}$ of R^2 with the Euclidean metric is

(a) an open set ( )

(b) a closed set ( )

(c) both open and closed ( )

(d) neither open nor closed ( )

4. Every singleton set in a discrete space is

(a) open ( )

(b) open and not closed ( )

(c) closed and not open ( )

(d) neither open nor closed ( )

5. A real-valued function continuous on a compact set is

(a) bounded above but not below ( )

(b) bounded below but not above ( )

(c) bounded and attains its bound ( )

(d) bounded and does not attain its bounds ( )

6. The image of a function continuous on a compact set is

(a) closed and bounded ( )

(b) neither closed nor bounded ( )

(c) closed but not bounded ( )

(d) bounded but not closed ( )

7. Which of the following functions does not possess first-order partial derivatives at the origin?

(a) $f(x, y) \quad \begin{cases} \dfrac{xy}{x^2 \quad 2y^2}, & (x, y) \quad (0, 0) \\ 0, & (x, y) \quad (0, 0) \end{cases}$ ( )

(b) $f(x, y) \quad \begin{cases} \dfrac{x^2 y}{x^4 \quad y^2}, & x^2 \quad y^2 \quad 0 \\ 0, & (x, y) \quad (0, 0) \end{cases}$ ( )

(c) $f(x, y) \quad \begin{cases} \dfrac{x^3 \quad y^3}{x^2 \quad y^2}, & (x, y) \quad (0, 0) \\ 0, & (x, y) \quad (0, 0) \end{cases}$ ( )

(d) $f(x, y) \quad \begin{cases} (x \quad 1)\sin \dfrac{1}{x \quad y}, & x \quad y \quad 0 \\ 0, & x \quad y \quad 0 \end{cases}$ ( )

8. If f_x exists throughout a neighbourhood of a point (a, b) and $f_y(a, b)$ exists, then for any point $(a \quad h, b \quad k)$ of this neighbourhood

(a) $f(a \quad h, b \quad k) \quad f(a, b) \quad hf_x(a \quad h, b \quad k) \quad k\{f_y(a, b) \quad \}$, where $0 \quad 1$ and is a function of k which tends to 0 with k ( )

(b) $f(a \quad h, b \quad k) \quad f(a, b) \quad hf_x(a \quad h, b \quad k) \quad k\{f_y(a, b) \quad \}$, where $0 \quad 1$ and is a function of k which tends to 0 with k ( )

(c) $f(a \quad h, b \quad k) \quad f(a, b) \quad hf_x(a \quad h, b \quad k) \quad k\{f_y(a, b) \quad \}$, where $0 \quad 1$ and is a function of k which tends to 0 with k ( )

(d) $f(a \quad h, b \quad k) \quad f(a, b) \quad hf_x(a \quad h, b \quad k) \quad k\{f_y(a, b) \quad \}$, where $0 \quad 1$ and is a function of k which tends to 0 with k ( )

9. Let $f : D \quad R, D \quad R^2$. Then $f_{xy}(a, b) \quad f_{yx}(a, b)$ if

(a) f_x is continuous and f_y merely exists at (a, b) ( )

(b) f_x and f_y are both continuous at (a, b) ( )

(c) f_x is continuous and f_y is differentiable at (a, b) ( )

(d) f_x and f_y are both differentiable at (a, b) ( )

10. Let $f : D \quad R, D \quad R^2$. Then $f(a, b)$ is a maximum value of f if for every point (x, y) of some neighbourhood of (a, b)

(a) $f(x, y) \quad f(a, b)$ is always positive ( )

(b) $f(x, y) \quad f(a, b)$ is always negative ( )

(c) $f(x, y) \quad f(a, b)$ is always zero ( )

(d) $f(x, y) \quad f(a, b)$ changes signs ( )

SECTION—B

(*Marks* : 15)

Each question carries 3 marks

1. *(a)* Prove that the intersection of a finite number of open sets is open but this does not hold for infinite sets in general.

OR

(b) Prove that the derived set of a set is closed.

2. *(a)* Prove that every closed subset of a compact metric space is compact.

OR

(b) Show that the derived set of every subset of a discrete space is empty.

3. *(a)* Prove that the image of a compact set under a continuous function is compact.

OR

(b) Prove that a function continuous on a compact set is uniformly continuous.

4. *(a)* Show that the function $f(x, y) \quad \sqrt{|xy|}$ is not differentiable at $(0, 0)$.

OR

(b) If $u^3 \quad v^3 \quad x \quad y$ and $u^2 \quad v^2 \quad x^2 \quad y^2$, then prove that

$$\frac{(u, v)}{(x, y)} \quad \frac{y \quad x}{3uv(u \quad v)}$$

5. *(a)* Prove that the function $f(x, y) \quad 2x^4 \quad 3x^2y \quad y^2$ has neither a minima nor a maxima at the origin.

OR

(b) Using Taylor's theorem, show that for $0 \qquad 1$

$$\sin x \sin y \quad xy \quad \frac{1}{6}\{(x^3 \quad 3xy^2)\cos \quad x \sin \quad y \quad (y^3 \quad 3x^2y)\sin \quad x \cos \quad y\}$$

(PART : B—DESCRIPTIVE)

(*Marks* : 50)

The figures in the margin indicate full marks for the questions

Answer **five** questions, selecting **one** from each Unit

UNIT—1

1. *(a)* Prove that the interior of a set S is the largest open subset of S. 3

(b) Define limit point of a set. Show with the help of an example that boundedness is not a necessary condition for a set to possess a limit point. 1+2=3

(c) Prove that a set is closed if and only if its complement is open. 4

2. *(a)* State Lindelof theorem and hence prove that a closed and bounded set is compact. 1+5=6

(b) If a sequence of closed intervals $[a_n, b_n]$ is such that

$$[a_{n \quad 1}, b_{n \quad 1}] \quad [a_n, b_n] \text{ and } \lim_{n} (a_n \quad b_n) \quad 0$$

then prove that there is one and only one point common to all the intervals. 4

UNIT—2

3. *(a)* Prove that every compact subset of a metric space is closed. 4

(b) Show that every closed sphere in a metric space is closed. 3

(c) Show that the closure of a subset of a metric space is closed. 3

4. *(a)* Let (X, d) be any metric space. Prove that a subset of X is closed if and only if its complement is open. 3

(b) Prove the completeness of the space R^n of all ordered n-tuples with the metric d, where $d(x, y) \quad \sqrt{\ {}_{i\ 1}^{n}(x_i \quad y_i)^2}, \quad x, y \quad R^n$. 7

UNIT—3

5. *(a)* Prove that a set is compact if and only if every infinite subset has a limit point in the set. 6

(b) Show that the function

$$f(x, y) \quad \begin{matrix} \dfrac{xy}{\sqrt{x^2 \quad y^2}}, & (x, y) \quad (0, 0) \\ 0 \quad , & (x, y) \quad (0, 0) \end{matrix}$$

is continuous at (0, 0). 4

6. *(a)* Let $f : D \quad R, D \quad R^n$, where D is a convex set. Show that f assumes every value between $f(x)$ and $f(y)$, $\quad x, y \quad D$. 5

(b) Prove that a function $f : D \quad R, D \quad R^n$ is continuous if and only if $f^{\ 1}(V)$ is closed in R for every closed set V in R. 5

UNIT—4

7. *(a)* Prove that a function which is differentiable at a point is continuous and possesses the first-order partial derivatives at that point but the converse is not necessarily true. 6

(b) Find the directional derivative of $f(x, y) \quad x^2 y \quad \sin x$ at the point $0, \overline{\ 2\ }$ in the direction of $\vec{v} \quad 3\hat{i} \quad 4\hat{j}$. 4

8. *(a)* If $\ ,\ ,\ $ are the roots of the equation in t such that $\dfrac{u}{a \quad t} \quad \dfrac{v}{b \quad t} \quad \dfrac{w}{c \quad t} \quad 1$, then prove that

$$\frac{(u, v, w)}{(\ ,\ ,\ )} \quad \frac{(\quad)(\quad)(\quad)}{(a \quad b)(b \quad c)(c \quad a)}$$ 5

(b) Prove that a function $f : D \quad R,\ D \quad R^2$ is differentiable at a point (a, b) of D if one of the partial derivatives is continuous at (a, b) and the other merely exists at (a, b). 5

UNIT—5

9. (a) State and prove Schwarz's theorem. 1+6=7

(b) Show that the conditions of Young's theorem are not satisfied for the function

$$f(x, y) \quad \begin{cases} \dfrac{x^2y^2}{x^2 \quad y^2}, & \text{when } (x, y) \quad (0, 0) \\ 0, & \text{when } (x, y) \quad (0, 0) \end{cases}$$

at $(0, 0)$ and $f_{xy}(0, 0) \quad f_{yx}(0, 0)$. 3

10. (a) State and prove Taylor's theorem for two variables. 1+3=4

(b) Examine the following functions for extreme values : 3+3=6

(i) $f(x, y) \quad 21x \quad 12x^2 \quad 2y^2 \quad x^3 \quad xy^2$

(ii) $f(x, y) \quad 4x^2 \quad xy \quad 4y^2 \quad x^3y \quad xy^3 \quad 4$

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