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(CBCS)

(6th Semester)

MATHEMATICS

TENTH PAPER

(Advanced Calculus)

Full Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

(SECTION : A—OBJECTIVE)

(*Marks* : 10)

Tick (✓) the correct answer in the brackets provided : 1×10=10

1. Regarding the condition of intergrability for a function f corresponding to the partition P of an interval $[a, b]$, the incorrect statement is

(a) $\lim_{(P)\quad 0} S(P, f) \qquad {}^{b}_{a} f\, dx$ ( )

(b) $\sup L(P, f) \quad \inf U(P, f)$ ( )

(c) $\lim_{(P)} \{U(P, f) \quad L(P, f)\} \quad 0$ ( )

(d) $\lim_{(P)\quad 0} \{U(P, f) \quad L(P, f)\} \quad 0$ ( )

2. For any two partitions P_1, P_2 of a bounded function f

(a) $L(P_1, f) \quad U(P_2, f)$ ( )

(b) $L(P_1, f) \quad U(P_2, f)$ ( )

(c) $L(P_2, f) \quad U(P_1, f)$ ( )

(d) $U(P_2, f) \quad L(P_1, f)$ ( )

3. If f and g be two positive functions such that $f(x) \quad g(x), \quad x \quad [a, b]$, then

(a) $\;{}_{a}^{b}\, g\,dx$ converges if $\;{}_{a}^{b}\, f\,dx$ converges ( )

(b) $\;{}_{a}^{b}\, f\,dx$ converges if $\;{}_{a}^{b}\, g\,dx$ converges ( )

(c) $\;{}_{a}^{b}\, f\,dx$ diverges if $\;{}_{a}^{b}\, g\,dx$ diverges ( )

(d) $\;{}_{a}^{b}\, f\,dx$ and $\;{}_{a}^{b}\, g\,dx$ behave alike ( )

4. If a function f is continuous on [0, 1], then the value of $\lim\limits_{n} \;{}_{0}^{1} \dfrac{nf(x)}{1 \quad n^2x^2}$ is

(a) $\dfrac{}{2} f(0)$ ( ) *(b)* $\dfrac{}{2} f(1)$ ( )

(c) $\dfrac{}{2} f(n)$ ( ) *(d)* $\dfrac{}{2} f(\ )$ ( )

5. The value of $\;{}_{0} \dfrac{\log(1 \quad a\cos x)}{\cos x}\,dx$ if $|a| \quad 1$ is

(a) $2 \quad \sin^{\;1} a$ ( )

(b) $\sin^{\;1} a$ ( )

(c) $2 \quad \sin^{\;1} a$ ( )

(d) None of the above ( )

6. The value of the improper integral $\;{}_{0}\, e^{\;x^2}\,dx$ is equal to

(a) $\dfrac{}{2}$ ( ) *(b)* $\dfrac{2}{}$ ( )

(c) $\dfrac{\sqrt{\ }}{2}$ ( ) *(d)* $\dfrac{2}{\sqrt{\ }}$ ( )

7. The value of the integral $\underset{C}{\;} x^2dx \quad xydy$ taken along the line segment from (1, 0) to (0, 1) is

(a) 0 ( )

(b) $\frac{1}{6}$ ( )

(c) 1 ( )

(d) $\frac{1}{6}$ ( )

8. The value of the double integral $x^2y^3\,dxdy$ over the circle $x^2 \quad y^2 \quad a^2$ is

(a) 0 ( )

(b) $\frac{1}{2}$ ( )

(c) $\frac{1}{2}$ ( )

(d) $\frac{\;}{2}$ ( )

9. By M_n-test, the sequence $\{f_n\}$ converges uniformly to f on $[a, b]$ if and only if

(a) $M_n \quad 0$ as $n \quad 0$ ( )

(b) $M_n \quad\quad$ as $n \quad 0$ ( )

(c) $M_n \quad \inf\{|f_n(x) \quad f(x)| : x \quad [a, b]\}$ ( )

(d) $M_n \quad 0$ as n ( )

10. The sequence of function $f_n(x) \quad nxe^{\;nx^2}$ is pointwise, but not uniformly convergent on

(a) $[0, k]$, where $k \quad 1$ ( )

(b) $[0, \quad]$ ( )

(c) $[0, \quad)$ ( )

(d) $(0, \quad)$ ( )

(SECTION : B—SHORT ANSWER)

(*Marks* : 15)

Answer the following questions : 3×5=15

UNIT—I

1. For the function defined by

$$f(x) \quad \begin{cases} 1\ , \text{ when } x \text{ is rational number} \\ 1\ , \text{ when } x \text{ is irrational number} \end{cases}$$

prove that $|f|$ is R-integrable on any interval $[a,\ b]$ while f is not integrable.

OR

For the integral $\displaystyle{}_{0}^{1}\ x\,dx$, find the upper Riemann integral corresponding to the division of [0, 1] into 6 equal intervals.

UNIT—II

2. Using Frullani integral, evaluate $\displaystyle{}_{0} \frac{\tan\ ^{1}(ax) \quad \tan\ ^{1}(bx)}{x}\,dx.$

OR

Prove that $\displaystyle{}_{0} \frac{\cos x}{x}\,dx$ is convergent at .

UNIT—III

3. If $f(x,\ y) \quad \dfrac{y^2}{x^2 \quad y^2}$ and $g(y) \quad {}_{0}^{1}\ f(x,\ y)\,dx$, then evaluate the value of

$${}_{0}^{1}\ f_y(x,\ 0)\,dx$$

OR

Evaluate $\displaystyle {}_0 \frac{dx}{a \quad b\cos x}$ if a is positive and $|b| \quad a$ and deduce that

$$ {}_0 \frac{dx}{(a \quad b\cos x)^2} \qquad \frac{a}{(a^2 \quad b^2)^{3/2}} $$

UNIT—IV

4. Find the value of line integral $\displaystyle {}_C \frac{ydx \quad xdy}{x^2 \quad y^2}$ around the circle $C : x^2 \quad y^2 \quad 1.$

OR

Evaluate the double integral $\displaystyle {}_0^2 {}_{x^2}^{x} y^2 xdydx.$

UNIT—V

5. Define pointwise and uniform convergence of a sequence of functions $\{f_n(x)\}$. Give an example to show the distinction between uniform and pointwise convergent.

OR

Prove that the sequence $f_n(x) \quad x^n$ is pointwise convergent on [0, 1] and evaluate the pointwise limit.

(SECTION : C—DESCRIPTIVE)

(*Marks* : 50)

Answer **one** question from each Unit

UNIT—I

1. *(a)* Show that $f(x) \quad |x|$ is integrable on $[\quad 1, 1]$ and hence evaluate $\displaystyle {}_1^1 f(x)\,dx.$ 5

(b) Prove that a necessary and sufficient condition for the integrability of a bounded function f is that to every $\quad 0$, there exists a corresponding $\quad 0$ such that

$$ U(P, f) \quad L(P, f) $$

for every partition P of $[a, b]$ with norm $\quad (P) \quad .$ 5

2. *(a)* Show that when $1 \quad x \quad 1$, then

$$\lim_{m} \quad {}_{0}^{x} \frac{t^{m}}{t \quad 1} dt \quad 0$$ 4

(b) If a function f is bounded and integrable on $[a, b]$, then prove that the integral function of f defined by $F(t) \quad {}_{a}^{t} f(x)\,dx,\ t \quad [a, b]$ is continuous on $[a, b]$. Furthermore, if f is continuous at a point $c \quad [a, b]$, then F is derivable at c and $F(c) \quad f(c)$. 6

UNIT—II

3. *(a)* Show that ${}_{1} \frac{\sin x}{x^{p}} dx$ converges absolutely for $p \quad 1$ but conditionally only for $0 \quad p \quad 1$. 5

(b) Examine the convergence of the following functions : 2½+2½=5

(i) ${}_{0}^{2} \frac{1}{(1 \quad x)^{2}} dx$

(ii) ${}_{2} \frac{2x^{2}}{x^{4} \quad 1} dx$

4. *(a)* Prove that the improper integral ${}_{a}^{b} f(x)\,dx$ is convergent at a if and only if to every $\quad 0$ there exists a corresponding $\quad 0$ such that

$$\left| {}_{a \quad 1}^{a \quad 2} f(x)\,dx \right| \quad \text{for } 0 \quad {}_{1}, \quad {}_{2}$$ 5

(b) Prove that the integral ${}_{1} x^{n \quad 1} e^{\quad x} dx$ is convergent if and only if $n \quad 0$. 5

UNIT—III

5. *(a)* Prove that uniformly convergent improper integral of a continuous function is itself continuous. 4

(b) If $a \quad b$, then show that

$$\overset{\overline{2}}{\underset{0}{}} \log \frac{1 \quad \cos \quad \cos x}{\cos x} \, dx \quad \frac{1}{2} \; \frac{^2}{4} \qquad {}^2$$ 6

6. *(a)* Examine the uniform convergence of the convergent improper integral

$$\overset{1}{\underset{1}{}} \frac{\cos yx}{\sqrt{1 \quad x^2}} \, dx \text{ in } (\quad , \quad)$$ 4

(b) Establish the right to integrate under integral sign for $\underset{0}{} \; e^{\quad xy} \cos mx \, dx$

and deduce that

$$\underset{0}{} \frac{e^{\quad ax} \quad e^{\quad bx}}{x} \cos mx \, dx \quad \frac{1}{2} \log \frac{b^2 \quad m^2}{a^2 \quad m^2} \;, \; a, b \quad 0$$ 6

UNIT—IV

7. *(a)* Evaluate $\underset{A}{} \; xy\,(x \quad y)\, dxdy$ over the area between $y \quad x^2$ and $y \quad x$. 4

(b) Prove that $\overset{1}{\underset{0}{}} \; \overset{1}{\underset{0}{}} \frac{(x \quad y)}{(x \quad y)^3} \, dy \quad dx \quad \frac{1}{2}$, but the value changes its sign as the order of integration interchange. 6

8. *(a)* Change the order of integration in the integral

$$\overset{1}{\underset{0}{}} dx \; \overset{\sqrt{1 \quad x^2}}{\underset{0}{}} \frac{e^y dy}{(1 \quad e^y)\sqrt{1 \quad x^2 \quad y^2}}$$

and hence evaluate it. 6

(b) Show that $\quad_{C}[(x \quad y)^3 dx \quad (x \quad y)^3 dy] \quad 3 \quad a^4$, where C is the circle $x^2 \quad y^2 \quad a^2$ in the counterclockwise sense. 4

UNIT—V

9. *(a)* Show by M_n-test that the sequence of function $f_n(x) \quad \frac{x}{1 \quad nx^2}, \quad x \quad R$ converges uniformly on any close interval *I*. 4

(b) State and prove Cauchy's criterion of uniform convergence of a sequence $\{f_n(x)\}$ of real valued functions on a set *E*. 1+5=6

10. *(a)* Examine the term by term integration of the series whose sum to first n terms is $n^2 x(1 \quad x)^n$, $0 \quad x \quad 1$. 5

(b) If the sequence $\{f_n(x)\}$ of continuous functions is uniformly convergent to a function f on $[a, b]$, then prove that f is also continuous. 5

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