 Student's Copy

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(CBCS)

(6th Semester)

MATHEMATICS

NINTH PAPER

(Modern Algebra)

Full Marks : 75

Time : 3 hours

(SECTION : A—OBJECTIVE)

(*Marks* : 10)

Each question carries 1 mark

Put a Tick ☑ mark against the correct answer in the boxes provided :

1. If the order of a group G with centre Z is p^n, where p is a prime number, then

(a) $Z = \{e\}$ ☐ (b) $Z \neq \{e\}$ ☐

(c) $Z = e$ ☐ (d) $Z = \phi$ ☐

2. The identity element of the quotient group G/H is

(a) e ☐ (b) 1 ☐

(c) H ☐ (d) e/H ☐

3. The necessary and sufficient conditions for a non-empty subset S of a ring R to be a subring are

(a) $a - b \in S$ and $ab \in S$ for all $a, b \in S$ ☐

(b) $a - b \in S$ and $a/b \in S$ for all $a, b \in S$ ☐

(c) $a + b \in S$ and $a/b \in S$ for all $a, b \in S$ ☐

(d) $a + b \in S$ and $ab \in S$ for all $a, b \in S$ ☐

4. An ideal S of the ring of integers I is maximal if S is generated by

(a) 5 ☐ *(b)* 9 ☐

(c) 12 ☐ *(d)* 15 ☐

5. A non-zero integer has

(a) no associates ☐

(b) exactly one associate ☐

(c) infinite number of associates ☐

(d) exactly two associates ☐

6. In the ring of integers, the greatest common divisors of 5 and 10 are

(a) 1 and 5 ☐ *(b)* 1 and −5 ☐

(c) 5 and −5 ☐ *(d)* −1 and 5 ☐

7. Which of the following statements is false?

(a) Every superset of a linearly dependent set of vectors is linearly dependent ☐

(b) There exists a basis for each finite dimensional vector space. ☐

(c) A system consisting of a single non-zero vector is always linearly independent. ☐

(d) Every linearly dependent subset of a finitely generated vector space $V(F)$ forms a part of a basis of V. ☐

8. The necessary and sufficient conditions for a vector space $V(F)$ to be a direct sum of its two subspaces U and W are

(a) $V = U + W$ and $U \cap W = \{0\}$ ☐

(b) $V = UW$ and $U \cap W = \{0\}$ ☐

(c) $V = U + W$ and $U \cap W \neq \{0\}$ ☐

(d) $V = U + W$ and $U \cap W = 0$ ☐

9. If a linear transformation $T : R^7 \to R^3$ has 4-dimensional kernel, then the dimension of range space of T is

(a) 3 ☐ (b) 10 ☐

(c) 4 ☐ (d) 11 ☐

10. Let T be a linear transformation from a vector space U into a vector space V, where dimension of U is $2n$, then Rank (T) + Nullity (T) =

(a) 1 ☐ (b) n ☐

(c) $2n$ ☐ (d) n^2 ☐

(SECTION : B—SHORT ANSWER)

(*Marks* : 15)

Each question carries 3 marks

Answer the following questions :

UNIT—I

1. If H is a subgroup of G and N is a normal subgroup of G, then show that $H \cap N$ is a normal subgroup of H.

OR

2. If the order of a group G is p^2, where p is a prime number, then prove that G is Abelian.

UNIT—II

3. Prove that every field is an integral domain.

OR

4. If F is a field, then prove that its only ideals are (0) and F itself.

UNIT—III

5. Find all the units of the integral domain of Gaussian integers.

OR

6. Prove that the necessary and sufficient condition for a non-zero element a in the Euclidean ring R to be a unit is that $d(a) = d(1)$.

UNIT—IV

7. In $V_3(R)$, where R is the field of real numbers, show that the set of vectors $\{(1, 2, 0), (0, 3, 1), (-1, 0, 1)\}$ is linearly independent.

OR

8. Show that the kernel of a homomorphism of a vector space is a subspace.

UNIT—V

9. Show that the function $T : R^2 \to R^2$ defined by $T(x, y) = (1 + x, y)$ is a linear transformation.

OR

10. Let $T : R^2 \to R^2$ be a linear map defined by $T(x, y) = (4x - 2y, 2x + y)$. Find the matrix representation of T relative to the ordered basis $\{(1, 1), (-1, 0)\}$.

(SECTION : C—DESCRIPTIVE)

(*Marks* : 50)

The figures in the margin indicate full marks for the questions

Answer **five** questions, taking **one** from each Unit

UNIT—I

1. *(a)* Prove that a subgroup H of a group G is a normal subgroup of G if and only if each left coset of H in G is a right coset of H in G. 4

(b) Prove that every homomorphic image of a group G is isomorphic to some quotient group of G. 6

2. *(a)* Define inner automorphism. Show that for an Abelian group, the only inner automorphism is the identity mapping whereas for non-Abelian groups, there exist non-trivial automorphisms. 1+3=4

(b) If H is a normal subgroup of a group G and K is a normal subgroup of G containing H, then prove that

$$G / K \cong (G / H) / (K / H)$$ 6

UNIT—II

3. *(a)* Prove that every finite integral domain is a field. 6

(b) Prove that the characteristic of an integral domain is either 0 or a prime number. 4

4. *(a)* Show that a commutative ring with unity is a field if it has no proper ideal. 4

(b) Prove that in a commutative ring R, an ideal S of R is prime if and only if the ring of residue classes R/S is an integral domain. 6

UNIT—III

5. *(a)* Define kernel of a ring homomorphism. Let f be a homomorphism of a ring R into a ring R'. Show that the kernel of f is an ideal of R. 1+4=5

(b) Let D be an integral domain with unity element 1. Show that two non-zero elements $a,\ b \in D$ are associates if and only if a/b and b/a. 5

6. *(a)* Let a and b be any two elements of a Euclidean ring R, not both of which are zero. Prove that a and b have a greatest common divisor d which can be expressed in the form

$$d = \lambda a + \mu b \text{ for some } \lambda, \mu \in R$$ 6

(b) If a is a prime element of a unique factorization domain R and b, c are any elements of R, then prove that

$$a/bc \Rightarrow a/b \text{ or } a/c$$ 4

UNIT—IV

7. *(a)* Prove that the union of two subspaces is a subspace if and only if one is contained in the other. 4

(b) If $V(F)$ is a finite dimensional vector space, then show that any two bases of V have the same number of elements. 6

8. *(a)* If U and W are two subspaces of a finite dimensional vector space $V(F)$, then prove that

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$$ 7

(b) Show that the vectors (1, 1, 2), (1, 2, 5), (5, 3, 4) do not form a basis of R^3. 3

UNIT—V

9. *(a)* Let V and W be vector spaces over the same field F and let T be a linear transformation from V into W. If V is finite dimensional, then prove that

$$\text{Rank } T + \text{Nullity } T = \dim V$$ 6

(b) Let $T : R^4 \to R^3$ be a linear transformation defined by

$$T(x, y, z, t) = (x - y + z + t,\ x + 2z - t,\ x + y + 3z - 3t)$$

Find the range space, null space, rank of T and nullity of T. 4

10. *(a)* Let V and U be vector spaces over the field K. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis of V. If $\{u_1, u_2, \ldots, u_n\}$ be a set of arbitrary vectors in U, then show that there exists a unique linear transformation $F: V \to U$ such that

$$F(v_j) = u_j \text{ for } j = 1, 2, \ldots, n$$ 6

(b) Find the matrix representation of linear map $T: R^3 \to R^3$ given by

$$T(x, y, z) = (x + z, -2x + y, -x + 2y + z)$$

relative to the basis $\{(1, 0, 1), (-1, 1, 1), (0, 1, 1)\}$. 4

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