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(CBCS)

(6th Semester)

MATHEMATICS

TWELFTH (B) PAPER

(Elementary Number Theory)

Full Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

(SECTION : A—OBJECTIVE)

(*Marks* : 10)

Tick (✓) the correct answer in the brackets provided : 1×10=10

1. The greatest common divisor of 1980 and 1617 is

(a) 33 ( )

(b) 1 ( )

(c) 17 ( )

(d) 11 ( )

2. The least common multiple of 2864 and 624 is

(a) 135231 ( )

(b) 6240 ( )

(c) 111696 ( )

(d) 2864 ( )

3. A solution of the linear congruence $13x \quad 5 (\mathrm{mod}\, 9)$ is

(a) 5 ( )

(b) 6 ( )

(c) 7 ( )

(d) 8 ( )

4. Which of the following is reduced residue system modulo 8?

(a) $\{1, 3, 5\}$ ( )

(b) $\{1, 2, 6\}$ ( )

(c) $\{1, 7\}$ ( )

(d) $\{1, 3, 5, 7\}$ ( )

5. The remainder when 3^{24} is divided by 5 is

(a) 1 ( )

(b) 2 ( )

(c) 3 ( )

(d) 4 ( )

6. If p is a positive prime and n is any positive integer, then

$$(1) \quad (p) \quad (p^2) \quad \cdots\cdots\cdots \quad (p^{n \;\; 1}) \quad (p^n)$$ is equal to

(a) p^n ( )

(b) $p^{n \;\; 1}$ ( )

(c) 1 ( )

(d) 0 ( )

7. Let p 17, a 3. Then the value of $\frac{a}{p}$ is

(a) 1 ( )

(b) –1 ( )

(c) 0 ( )

(d) 2 ( )

8. If p denotes an odd prime, then the Legendre symbol $\frac{a}{p}$ is defined to be 1 if

(a) $p|a$ ( )

(b) a is a quadratic non-residue ( )

(c) a is a quadratic residue ( )

(d) None of the above ( )

9. The exponent of 7 in 1000! is

(a) 100 ( )

(b) 7 ( )

(c) 164 ( )

(d) 165 ( )

10. If n 71, then the value of (n) is

(a) 71 ( )

(b) 72 ( )

(c) 73 ( )

(d) 74 ( )

(SECTION : B—SHORT ANSWER)

(*Marks* : 15)

Answer the following : 3×5=15

UNIT—I

1. If $a|b$ and $b|c$, prove that $a|c$.

OR

2. If $p, q_1, q_2, q_3, \cdots\cdots, q_n$ are all primes and $p|q_1q_2q_3 \cdots\cdots q_n$, prove that $p \quad q_k$ for some k, where $1 \quad k \quad n$.

UNIT—II

3. Prove that the number of primes is infinite.

OR

4. If $a \quad b(\text{mod } m_1)$, $a \quad b(\text{mod } m_2)$ and m is the least common multiple of m_1 and m_2, prove that $a \quad b(\text{mod } m)$.

UNIT—III

5. If p is a positive prime and n is any positive integer, find the value of the series $(1) \quad (p) \quad (p^2) \quad \cdots\cdots \quad (p^{n \quad 1}) \quad (p^n)$

OR

6. If $\gcd(a, m) \quad 1$ and a is of order $\quad (n)$ modulo n, prove that a is a primitive root of n.

UNIT—IV

7. Let p be an odd prime. Prove that

$$\frac{a}{p} \quad \frac{b}{p} \qquad \frac{ab}{p}$$

OR

8. Find all odd primes p such that $x^2 \quad 13(\text{mod } p)$ has a solution.

UNIT—V

9. Let x be any real number. Prove that $\frac{[x]}{m} \quad \frac{x}{m}$.

OR

10. Prove that $\frac{(n)}{n} \quad\quad \frac{(d)}{d}$.

(SECTION : C—DESCRIPTIVE)

(*Marks* : 50)

Answer the following : 10×5=50

UNIT—I

1. *(a)* If g is the greatest common divisor of b and c, prove that there exists integers x_0 and y_0 such that $g \quad (b, c) \quad bx_0 \quad cy_0$. 5

(b) Find the number of distinct positive integral divisors and their sum for the integer 4800. 5

OR

2. *(a)* If $a|c$, $b|c$ and $(a, b) \quad 1$, show that $ab|c$. Also show that the conclusion is false when $(a, b) \quad 1$. 5

(b) Using the Euclidean algorithm obtain integers x and y satisfying the condition that $\gcd(42823, 6409) \quad 42823x \quad 6409y$. 5

UNIT—II

3. *(a)* If a and b are two integers, prove that $a \quad b(\bmod m)$ if and only if a and b have the same remainder when divided by m. 5

(b) Solve the congruence

$$235x \quad 54(\bmod 7)$$ 5

OR

4. *(a)* Let m be a fixed positive integer and $S \quad \{0, 1, 2, 3, \cdots \cdots, m \quad 1\}$. Prove that no two integers of S are congruent modulo m to each other and every $x \quad Z$ is congruent modulo to one of the integers of S. 6

(b) If $a \quad b(\bmod m)$, prove that $a \quad x \quad b \quad x(\bmod m)$ and $ax \quad bx(\bmod m)$ for all $x \quad Z$. 4

UNIT—III

5. *(a)* Prove that $28! \quad 233 \quad 0 \pmod{899}$ using Wilson's theorem. 4

(b) Verify that 2 is a primitive root of 19, but not of 17. 3

(c) If p is a positive prime and a is any integer, prove that $a^p \quad a \pmod{p}$. 3

OR

6. *(a)* State and prove Euler's theorem. 1+5=6

(b) If $n \quad 1$, prove that the sum of all positive integers which are less than n and prime to n is $\frac{1}{2} n \quad (n)$. 4

UNIT—IV

7. *(a)* If Q is odd and $Q \quad 0$, prove that

$$\frac{1}{Q} \quad (\quad 1)^{\frac{(Q \quad 1)}{2}} \quad \text{and} \quad \frac{2}{Q} \quad (\quad 1)^{\frac{(Q^2 \quad 1)}{8}}$$ 5

(b) Using Chinese remainder theorem, find the least positive integer x which satisfies

$$x \quad 2 \pmod{3}$$
$$x \quad 4 \pmod{5}$$
$$x \quad 5 \pmod{7}$$ 5

OR

8. *(a)* Let p be a prime and integer $n \quad 0$. Let $f(x) \quad a_n x^n \quad a_{n \quad 1} x^{n \quad 1} \quad \cdots \quad a_0$ be a polynomial of degree n modulo p. Prove that the congruence $f(x) \quad 0 \pmod{p}$ has at most n mutually incongruent solutions modulo p. 5

(b) Find the solutions of polynomial congruence

$$f(x) \quad x^2 \quad 7x \quad 2 \quad 0 \pmod{5^3}$$ 5

UNIT—V

9. *(a)* State and prove Mobius inversion formula. 6

(b) Let x be any real number. Prove that $[x] \quad [y] \quad [x \quad y] \quad [x] \quad [y] \quad 1$. 4

OR

10. *(a)* If f is a multiplicative arithmetic function and F is denoted by $F(n) \quad {}_{d|n} f(d)$, prove that F is also multiplicative. 4

(b) Find the three different Pythagorean triples not necessarily primitive of the form 16, y, z. 3

(c) Find the general solution of $10x \quad 8y \quad 42$. 3

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