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(CBCS)

(6th Semester)

MATHEMATICS

TENTH PAPER

(Advanced Calculus)

Full Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

(SECTION : A—OBJECTIVE)

(*Marks* : 10)

Put a Tick ☑ mark against the correct answer in the boxes provided : 1×10=10

1. If f is integrable on $[a, b]$, then $|f|$ is integrable and

(a) $\left|\int_a^b f\ dx\right| \quad \int_a^b |f|\,dx$ ☐

(b) $\left|\int_a^b f\ dx\right| \quad \int_a^b |f|\,dx$ ☐

(c) $\left|\int_a^b f\ dx\right|$ and $\int_a^b |f|\,dx$ are always equal ☐

(d) None of the above ☐

2. For a bounded and integrable function f, if $b \quad a$, then

(a) $m(b \quad a) \quad {}_a^b f\,dx \quad M(b \quad a)$ ☐

(b) $m(b \quad a) \quad {}_a^b f\,dx \quad M(b \quad a)$ ☐

(c) $m(b \quad a) \quad {}_a^b f\,dx \quad M(b \quad a)$ ☐

(d) $m(b \quad a) \quad {}_a^b f\,dx \quad M(b \quad a)$ ☐

3. The improper integral ${}_0 \frac{1}{x^n}\,dx$ is convergent if and only if

(a) $n \quad 1$ ☐

(b) $n \quad 1$ ☐

(c) $n \quad 1$ ☐

(d) None of the above ☐

4. If f and g are two positive functions on $[a, b]$ such that

$$\lim_{x \quad a} \frac{f(x)}{g(x)} \quad l$$

where l is a non-zero finite number, then

(a) ${}_a^b f\,dx$ converges if ${}_a^b g\,dx$ converges ☐

(b) ${}_a^b g\,dx$ converges if ${}_a^b f\,dx$ diverges ☐

(c) ${}_a^b f\,dx$ converges if ${}_a^b g\,dx$ diverges ☐

(d) ${}_a^b g\,dx$ converges but does not need to imply ${}_a^b f\,dx$ converges ☐

5. Suppose f is continuous function of two variables with domain as rectangle $[a, b; c, d]$ $\mathbb{R}^2$. Then the function (y) ${}_a^b f(x, y)dx$ for a fixed value of y $[c, d]$ is

(a) discontinuous in $[c, d]$ ☐

(b) derivable even though f_y does not exist and continuous ☐

(c) continuous in $[c, d]$ ☐

(d) None of the above ☐

6. The value of the integral ${}_0 \frac{\tan^{\ 1} ax}{x(1 \quad x)^2} dx$, if a 0, is

(a) $\frac{}{2}$ ☐ (b) $\frac{}{2}$ ☐

(c) $\frac{}{2(1 \quad a)}$ ☐ *(d)* $\frac{}{2}\log(1 \quad a)$ ☐

7. The value of the integral ${}_C x^2 y^3 dxdy$ over the circle C, x^2 y^2 1 is

(a) $\frac{1}{6}$ ☐ (b) 0 ☐

(c) $\frac{1}{6}$ ☐ (d) log 2 ☐

8. The integral ${}_C xy\,dx$ along the arc of a parabola x y^2 from (1, 1) to (1, 1) is

(a) 3 ☐ *(b)* $\frac{2}{3}$ ☐

(c) 2 ☐ *(d)* $\frac{4}{5}$ ☐

9. The sequence of function $\{f_n\}$, where $f_n(x) \quad x^n$ is

(a) neither pointwise nor uniformly continuous for $x \quad [0, 1]$ ☐

(b) pointwise continuous for $x \quad [0, 1]$ ☐

(c) uniformly continuous for $x \quad [0, 1]$ ☐

(d) None of the above ☐

10. The series $\displaystyle \sum_{n \quad 1} \frac{1}{n^p}$ is uniformly continuous if

(a) $p \quad 1$ ☐

(b) $p \quad 1$ ☐

(c) $p \quad 1$ ☐

(d) $p \quad 1$ ☐

(SECTION : B—SHORT ANSWER)

(*Marks* : 15)

Answer the following questions : 3×5=15

UNIT—I

1. Show that if P is a refinement of P, then for a bounded function f, $U(P \;, f) \quad U(P, f)$.

OR

Compute $U(P, f)$ and $L(P, f)$ for the function $f(x) \quad x, 0 \quad x \quad 1$ on taking the partition

$$P \quad 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1 \quad \text{of } [0,1]$$

UNIT—II

2. Show that

$$\int_0^1 \frac{dx}{x^2}$$

is not convergent.

OR

Examine the convergence of the improper integral

$$\int_0^1 \frac{x^2}{\sqrt{x^5 \quad 1}}\, dx$$

UNIT—III

3. Evaluate $\int_0 \frac{\cos(ax) \quad \cos(bx)}{x}\, dx$

OR

Examine the uniform convergence of the improper integral

$$\int_1^1 \frac{\cos x}{\sqrt{1 \quad x^2}}\, dx \text{ in } (\quad , \quad)$$

UNIT—IV

4. Evaluate using Green's theorem $\int_C (xdx \quad x^2y^2dy)$, where C is the triangle with vertices (0, 0), (0, 1) and (1, 1) oriented positively.

OR

Evaluate $\int_C (x^2 \quad y^2)dx$, where C is the arc of the parabola $y^2 \quad 4ax$ between (0, 0) to $(a, 2a)$.

UNIT—V

5. Show that the sequence $\{f_n\}$, where $f_n(x) \quad \frac{x}{1 \quad nx^2}, x \quad \mathbb{R}$, converges uniformly on any closed interval.

OR

What do you mean by the pointwise convergence and the uniform convergence of a sequence of real-valued functions?

(SECTION : C—DESCRIPTIVE)

(*Marks* : 50)

Answer the following questions : 10×5=50

UNIT—I

1. *(a)* Prove that if f and g are two bounded and integrable functions on $[a, b]$, then their product fg is also bounded and integrable on $[a, b]$. 5

(b) Prove that if a function f is monotonic on $[a, b]$, then it is integrable on $[a, b]$. 5

OR

2. *(a)* Prove that if f is monotonic and f, f and g are all continuous in $[a, b]$, then there exists $t \quad [a, b]$ such that

$$ {}_a^b f(x)\, g(x)\, dx \quad f(a) \; {}_a^t g(x)\, dx \quad f(b) \; {}_t^b g(x)\, dx $$ 5

(b) Show that x^2 is integrable on any interval $[0, k]$. 5

UNIT—II

3. *(a)* Prove that every absolute convergent integral in $[a, b]$ is convergent in $[a, b]$. 5

(b) Examine the convergence of

$$\int_0^1 \frac{dx}{x^{3/2}(1 \quad x)^{1/2}}$$ 5

OR

4. *(a)* If f and g are two positive functions on $[a, b]$ such that

$$\lim_{x \to a} \frac{f(x)}{g(x)} \quad l$$

where l is non-zero finite number, then show that the two integrals $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ converge and diverge together at a. 5

(b) Evaluate : 2½+2½=5

(i) $\int_1^2 \frac{xdx}{\sqrt{x \quad 1}}$

(ii) $\int_1 \frac{dx}{\sqrt{1 \quad x^3}}$

UNIT—III

5. *(a)* Let $f(x, y)$ be a continuous function in $R[a, b; c, d]$. Then show that $(y) \quad \int_a^b f(x, y)dx$ is continuous in $[c, d]$. 5

(b) If $|a| \quad 1$, then show that

$$\int_0 \frac{\log(1 \quad a\cos x)}{\cos x} dx \qquad \sin^{1} a$$ 5

OR

6. *(a)* If f is a continuous function when $c \le y \le d$, $x \ge a$; and the integral $\phi(y) = \int_a^\infty f(x, y)dx$ is uniformly convergent, then show that ϕ can be integrated under the integral sign, i.e.,

$$\int_c^d \left[\int_a^\infty f(x,y)\,dx\right] dy = \int_c^d \phi(y)\,dy = \int_a^\infty \left[\int_c^d f(x,y)dy\right] dx$$ 5

(b) Show that

$$\int_0^\infty \frac{\cos yx}{1+x^2}\,dx = \frac{\pi}{2}e^{-y}$$ 5

UNIT—IV

7. *(a)* Find the value of $\oint_C (x+y^2)\,dx + (x^2+y)\,dy$ taken along the clockwise sense along the closed curve formed by $y^2 = x$ and $y = x$. 5

(b) Evaluate $\iint_R f(x,y)\,dx\,dy$ over a rectangle $R = [0, 1; 0, 1]$, where

$$f(x, y) = \begin{cases} x+y & ; \quad \text{if } x^2 \le y \le 2x^2 \\ 0 & ; \quad \text{elsewhere} \end{cases}$$ 5

OR

8. *(a)* Change the order of integration and evaluate

$$\int_0^\infty \int_0^y \frac{e^{-y}}{y}\,dx\,dy$$ 5

(b) Verify Green's theorem for $\oint_C x^2y\,dx + xy^2\,dy$, where C is taken along the first quadrant from the closed path bounded by $y = x$ and $x^2 = y^3$. 5

UNIT—V

9. Prove that the necessary and sufficient condition for uniform convergence of a sequence of the function $\{f_n(x)\}$ on a domain $[a, b]$ is that for every $\quad 0$ there exists a ve integer n such that

$$|f_{n \quad p}(x) \quad f_n(x)| \quad , \quad n \quad m, x \quad [a, b] \text{ and } \quad p, m \quad \mathbb{N}$$

10

OR

10. *(a)* Apply Cauchy criterion of convergence to show that the sequence of functions $\{f_n(x)\}$ converges uniformly on $[0, 1]$, where

$$f_n(x) \quad \frac{n \quad p}{x \quad n \quad p}$$

5

(b) Consider an infinite series of the form $_{n \quad 1}\frac{x}{n(n \quad 1)}$. Determine whether it is pointwise and also uniformly convergent for $x \quad [0, k]$. 5

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