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(CBCS)

(6th Semester)

MATHEMATICS

NINTH PAPER

(Modern Algebra)

Full Marks : 75

Time : 3 hours

(SECTION : A—OBJECTIVE)

(*Marks* : 10)

Each question carries 1 mark

Put a Tick ☑ mark against the correct answer in the box provided :

1. If a and b be two elements of a group G, then b is conjugate to a if

(a) $b \quad a^{1}xa; \ x \quad G$ ☐

(b) $b \quad axa^{1}; \ x \quad G$ ☐

(c) $b \quad x^{1}ax; \ x \quad G$ ☐

(d) $b \quad xa^{1}x; \ x \quad G$ ☐

2. If G is a group and a is a fixed element of G, then the mapping $f_a : G \quad G$ is an inner automorphism if

(a) $f_a(x) \quad a^{\;1}xa \quad x \quad G$ ☐

(b) $f_a(x) \quad ax^{\;1}a \quad x \quad G$ ☐

(c) $f_a(x) \quad x^{\;1}ax \quad x \quad G$ ☐

(d) $f_a(x) \quad xax^{\;1} \quad x \quad G$ ☐

3. The necessary and sufficient conditions for a non-empty subset K of a field F to be a subfield of F are

(a) $a \quad b \quad K$ and $ab \quad K$ for all $a, b \quad K$ ☐

(b) $a \quad b \quad K$ and $ab^{\;1} \quad K$ for all $a, b \quad K, b \quad 0$ ☐

(c) $a \quad b \quad K$ and $ab \quad K$ for all $a, b \quad K$ ☐

(d) $a \quad b \quad K$ and $ab^{\;1} \quad K$ for all $a, b \quad K, b \quad 0$ ☐

4. The characteristic of the ring $(I_6, \quad _6, \quad _6)$, where $I_6 \quad \{0, 1, 2, 3, 4, 5\}$ is

(a) 3 ☐

(b) 5 ☐

(c) 0 ☐

(d) 6 ☐

5. The associates of a non-zero element $a \quad ib$ of the ring of Gaussian integers $D \quad \{a \quad ib, a, b \quad I\}$ are

(a) $a \quad ib, \quad a \quad ib, \quad b \quad ia, b \quad ia$ ☐

(b) $a \quad ib, a \quad ib, \quad a \quad ib, \quad a \quad ib$ ☐

(c) $a \quad ib, \quad a \quad ib, b \quad ia, b \quad ia$ ☐

(d) $a \quad ib, \quad a \quad ib, \quad b \quad ia, b \quad ia$ ☐

6. The necessary and sufficient condition for a non-zero element a in the Euclidean ring R to be a unit is that

(a) $d(a) \quad d(1)$ ☐

(b) $d(a) \quad d(1)$ ☐

(c) $d(a) \quad d(1)$ ☐

(d) $d(a) \quad d(1)$ ☐

7. Which of the following sets of vectors is linearly dependent?

(a) $\{(1, 2, 0), (0, 3, 1), (\quad 1, 0, 1)\}$ ☐

(b) $\{(\quad 1, 2, 1), (3, 0, \quad 1), (\quad 5, 4, 3)\}$ ☐

(c) $\{(2, 1, 4), (1, \quad 1, 2), (3, 1, \quad 2)\}$ ☐

(d) $\{(2, \quad 3, 1), (3, \quad 1, 5), (1, \quad 4, 3)\}$ ☐

8. If W be a subspace of a finite dimensional vector space $V(F)$, then

(a) $\dim V / W \quad \dim V \quad \dim W$ ☐

(b) $\dim V / W \quad \dim V \quad \dim W$ ☐

(c) $\dim V / W \quad \dim W \quad \dim V$ ☐

(d) $\dim V / W \quad \dim V \quad \dim W$ ☐

9. Let V be a vector space over the field F. A function T satisfying

$$T(a \quad b \ ) \quad aT(\ ) \quad bT(\ )$$

for all $\ , \quad V$ and for all $a, b \quad F$ is called a linear functional if

(a) T is a function from V into V ☐

(b) T is a function from F into F ☐

(c) T is a function from V into F ☐

(d) T is a function from F into V ☐

10. Let V and U be vector spaces over F of dimensions m and n respectively. If $T : U \quad V$ is a linear map of rank r, then the dimension of kernel of T is

(a) $r \quad m$ ☐

(b) $m \quad r$ ☐

(c) $n \quad r$ ☐

(d) $r \quad n$ ☐

(Section : B—short answer)

(*Marks* : 15)

Each question carries 3 marks

Answer the following questions :

Unit—I

1. Prove that the intersection of any two normal subgroups of a group is a normal subgroup.

OR

2. Show that every quotient group of an Abelian group is Abelian and the converse is not true.

Unit—II

3. Prove that a skew field has no divisors of zero.

OR

4. Show that every ideal of a ring R is a subring of R but the converse is not necessarily true.

Unit—III

5. Let f be a homomorphism of a ring R into a ring R . Show that the kernel of f is an ideal of R.

OR

6. Show that every Euclidean ring possesses unity element.

Unit—IV

7. Show that the set $W \quad \{(x, y, z) : x \quad 3y \quad 4z \quad 0\}$ is a subspace of R^3.

OR

8. If $V(F)$ is a finite dimensional vector space of dimension n, then show that any set of n linearly independent vectors in V forms a basis of V.

UNIT—V

9. Show that the function $T : R^2 \quad R^2$ defined by $T(x, y) \quad (x^2,\ y)$ is not a linear transformation.

OR

10. Let $T : R^3 \quad R^3$ be a linear transformation defined by

$$T(x, y, z) \quad (3x \quad z, \quad 2x \quad y, \quad x \quad 2y \quad 4z)$$

Compute the matrix A of T with respect to the standard basis of R^3.

(SECTION : C—DESCRIPTIVE)

(*Marks* : 50)

The figures in the margin indicate full marks for the questions

Answer **five** questions, taking **one** from each Unit

UNIT—I

1. *(a)* Prove that a subgroup H of a group G is a normal subgroup of G if and only if the product of two right cosets of H in G is again a right coset of H in G. 5

(b) Define centre of a group. Prove that the centre Z of a group G is a normal subgroup of G. 1+4=5

2. *(a)* State the fundamental theorem on homomorphism of groups. Let H be any subgroup of a group G and if N is any normal subgroup of G, then prove that

$$\frac{HN}{N} \quad \frac{H}{H \quad N}$$ 1+6=7

(b) Show that for an Abelian group, the only inner automorphism is the identity mapping whereas for non-Abelian groups, there exists non-trivial automorphisms. 3

UNIT—II

3. *(a)* Prove that a ring R is without zero divisors if and only if the cancellation laws hold in R. 4

(b) Prove that a finite commutative ring without zero divisors is a field. 6

4. *(a)* If F is a field, then prove that its only ideals are (0) and F itself. 3

(b) Prove that an ideal S of a commutative ring R with unity is maximal if and only if the residue class ring R/S is a field. 7

UNIT—III

5. *(a)* Let D be an integral domain with unity element 1. Show that two non-zero elements $a, b \quad D$ are associates if and only if $a|b$ and $b|a$. 5

(b) Define Euclidean ring. Show that every field is a Euclidean ring. 1+4=5

6. *(a)* Let R be a Euclidean ring and a, b be two non-zero elements in R, then prove that—

(i) if b is a unit in R, $d(ab) \quad d(a)$

(ii) if b is not a unit in R, $d(ab) \quad d(a)$ 2+4=6

(b) If a is a prime element of a unique factorization domain R and b, c are any elements of R, then prove that $a|bc \quad a|b$ or $a|c$. 4

UNIT—IV

7. *(a)* Define linearly dependent set of vectors. Show that if two vectors are linearly dependent, one of them is a scalar multiple of the other. 1+3=4

(b) Prove that every linearly independent subset of a finitely generated vector space $V(F)$ is either a basis of V or can be extended to form a basis of V. 6

8. *(a)* If U and W are two subspaces of a finite dimensional vector space $V(F)$, then prove that $\dim(U \quad W) \quad \dim U \quad \dim W \quad \dim(U \quad W)$. 7

(b) Show that the kernel of a homomorphism of a vector space is a subspace. 3

UNIT—V

9. *(a)* Let U be an n-dimensional vector space over the field F and let V be an m-dimensional vector space over F. Then prove that the vector space $L(U, V)$ of linear transformations from U into V is also finite dimensional and is of dimension mn. 6

(b) Let $T : R^3 \quad R^3$ be a linear transformation defined by

$$T(x, y, z) \quad (x \quad 2y \quad z, y \quad z, x \quad y \quad 2z)$$

Find the range space, null space, rank of T and nullity of T. 4

10. *(a)* Let V and W be vector spaces over the field F and let T be a linear transformation from V into W. If V is finite dimensional, then prove that $\text{rank}T \quad \text{nullity}T \quad \dim V$. 6

(b) Find the matrix representation of the linear map $T : R^3 \quad R^3$ given by

$$T(x, y, z) \quad (z, y \quad z, x \quad y \quad z)$$

relative to the basis $\{(1, 0, 1), (\quad 1, 2, 1), (2, 1, 1)\}$. 4

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