**2 0 2 3**

( CBCS )

( 4th Semester )

# MATHEMATICS

FOURTH PAPER

**( Vector Calculus and Solid Geometry )**

*Full Marks* : 75

*Time* : 3 hours

*The figures in the margin indicate full marks for the questions*

## ( SECTION : A—OBJECTIVE )

( *Marks* : 10 )

Tick (✓) the correct answer in the brackets provided : 1×10=10

**1.** The area of the triangle *ABC* formed by the points $A(1, 2, 3)$, $B(2, 3, 1)$ and $C(3, 1, 2)$ is

*(a)* $\frac{3}{2}$ ( )

*(b)* $\frac{\sqrt{3}}{2}$ ( )

*(c)* $\frac{3\sqrt{3}}{2}$ ( )

*(d)* $\frac{9}{2}$ ( )

**2.** The projection of the vector $\hat{i} - 2\hat{j} + \hat{k}$ on the vector $4\hat{i} - 4\hat{j} + 7\hat{k}$ is

*(a)* 81 ( )

*(b)* $\frac{19}{9}$ ( )

*(c)* 19 ( )

*(d)* 7 ( )

**3.** If $\vec{r} = \cos\omega t\,\vec{a} + \sin\omega t\,\vec{b}$, then $\vec{r} \times \frac{d\vec{r}}{dt}$ is equal to

*(a)* $\vec{a} \times \vec{b}$ ( )

*(b)* $\omega^2\vec{r}$ ( )

*(c)* $\omega\vec{a} \times \vec{b}$ ( )

*(d)* $\frac{d^2\vec{r}}{dt^2}$ ( )

**4.** $\int\left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right)dt$ is equal to

*(a)* $\vec{r}$ ( )

*(b)* $\frac{d\vec{r}}{dt}$ ( )

*(c)* $\vec{r} \cdot \frac{d\vec{r}}{dt}$ ( )

*(d)* $\vec{r} \times \frac{d\vec{r}}{dt}$ ( )

**5.** The equation $ax^2 - 12xy + 9y^2 + 20x - 2fy - 11 = 0$ will represent a pair of parallel lines if

*(a)* $a = 4, \ f = 15$ ( )

*(b)* $a = -6, \ f = -10$ ( )

*(c)* $a = -6, \ f = 9$ ( )

*(d)* $a = 2, \ f = 6$ ( )

**6.** The conic $2x^2 - 3xy + y^2 - 5x + 4y - 2 = 0$ represents

*(a)* parabola ( )

*(b)* hyperbola ( )

*(c)* ellipse ( )

*(d)* a pair of straight lines ( )

**7.** The equation of a plane through $P(a, b, c)$ and perpendicular to $OP$, where $O$ is origin, is

*(a)* $ax + by + cz = 1$ ( )

*(b)* $ax + by + cz = a^2 + b^2 + c^2$ ( )

*(c)* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ ( )

*(d)* $x + y + z = a + b + c$ ( )

**8.** The angle between the planes represented by $6x^2 + 4y^2 - 10z^2 + 3yz + 4zx - 11xy = 0$ is

(a) $\pi$ ( )

(b) $\tan^{-1}\left(-\frac{3}{4}\right)$ ( )

(c) $\frac{\pi}{2}$ ( )

(d) $\frac{\pi}{4}$ ( )

**9.** The axis of the cylinder $f(x, y) = 0$, whose guiding curve is $f(x, y) = 0$, $z = 0$ is

(a) $X$-axis ( )

(b) $Y$-axis ( )

(c) $Z$-axis ( )

(d) not defined ( )

**10.** The cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ will have three mutually perpendicular tangent planes if

(a) $a^2 + b^2 + c^2 - fg - gh - hf = 0$ ( )

(b) $ab + bc + ca = f^2 + g^2 + h^2$ ( )

(c) $a^2 + b^2 + c^2 = 3fgh$ ( )

(d) $f^2 + g^2 + h^2 = 3abc$ ( )

## ( SECTION : B—SHORT ANSWER )

( *Marks* : 15 )

Answer the following : 3×5=15

### UNIT—I

**1.** If $\hat{a}$ and $\hat{b}$ are unit vectors and $\theta$ is the angle between them, then show that $\sin\frac{\theta}{2} = \frac{1}{2}|\hat{a} - \hat{b}|$.

**OR**

**2.** Find the unit tangent vector for the plane curve $\vec{r} = t\hat{i} + (\log\cos t)\,\hat{j}$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

### UNIT—II

**3.** Show that $\hat{r} \times d\hat{r} = \frac{\vec{r} \times d\vec{r}}{r^2}$, where $\vec{r} = r\,\hat{r}$.

**OR**

**4.** If $V$ is the volume enclosed by a closed surface $S$ and $\vec{F} = x\hat{i} + 2y\hat{j} + 3z\hat{k}$, then find the value of $\int_S \vec{F} \cdot \vec{n}\, dS$.

### UNIT—III

**5.** Find the values of $a$ and $g$ for which the curve $ax^2 + 8xy + 4y^2 + 2gx + 4y + 1 = 0$ represents a conic having infinitely many centres.

**OR**

**6.** Show that the conic passing through the point of intersection of two rectangular hyperbolas is also a rectangular hyperbola.

UNIT—IV

**7.** Find the plane that bisects the acute angle between the planes $2x - y + 2z + 3 = 0$ and $3x - 2y + 6z + 8 = 0$.

**OR**

**8.** Find the equation of the plane containing the line $\frac{x+2}{2} = \frac{y+3}{3} = \frac{z-4}{-2}$ and passing through the point (0, 6, 0).

UNIT—V

**9.** Find the equation of sphere passing through the points (0, − 2, − 4), (2, − 1, − 1) and whose centre lies on the line $5y + 2z = 0 = 2x - 3y$.

**OR**

**10.** Show that the equation $ax^2 + by^2 + cz^2 + 2fx + 2gy + 2hz + d = 0$ will represent a cone if $\frac{f^2}{a} + \frac{g^2}{b} + \frac{h^2}{c} = d$.

## ( SECTION : C—DESCRIPTIVE )

( *Marks* : 50 )

Answer the following : 10×5=50

UNIT—I

**1.** *(a)* Find the set of vectors reciprocal to the vectors $\vec{a}$, $\vec{b}$ and $\vec{a} \times \vec{b}$.

*(b)* Prove that $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}\ \vec{c}\ \vec{d}\,]\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}\,]\vec{a}$.

**OR**

**2.** *(a)* If a particle is moving along the curve $\vec{r}(t) = 20t\hat{i} + 5t^2\hat{j}$, then find its tangential and normal components of acceleration at time $t = 2$.

*(b)* Prove that, if the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are non-coplanar vectors, then $\vec{a} + \vec{b}$, $\vec{b} + \vec{c}$, $\vec{c} + \vec{a}$ are also non-coplanar vectors and $[\vec{a} + \vec{b}\ \ \vec{b} + \vec{c}\ \ \vec{c} + \vec{a}\,] = 2[\vec{a}\ \vec{b}\ \vec{c}\,]$.

## UNIT—II

**3.** *(a)* If $\vec{r} \times d\vec{r} = \vec{0}$, then prove that $\hat{r}$ = constant vector.

*(b)* If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then find the value of $\text{grad}(\log|\vec{r}|)$.

**OR**

**4.** *(a)* Evaluate $\text{div}\left(\frac{\vec{r}}{r}\right)$, where $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$.

*(b)* Evaluate $\int_S (y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}) \cdot \hat{n}\, dS$, where $S$ is the part of the sphere $x^2 + y^2 + z^2 = 1$ above the $XY$-plane and bounded by this plane.

## UNIT—III

**5.** *(a)* If any change of axes without changing of origin the quantity $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ transforms to $a'x'^2 + 2h'x'y' + b'y'^2 + 2g'x' + 2f'y' + c' = 0$, then show that the expressions $(ab - h^2)$ and $(f^2 + g^2)$ are invariant.

*(b)* Prove that the equation $y^3 - x^3 + 3xy(y - x) = 0$ represents three straight lines equally inclined to one another.

**OR**

**6.** *(a)* Show that the four normals can be drawn to an ellipse through a given point and the feet of the normals lie on a rectangular hyperbola.

*(b)* Prove that the locus of the middle points of focal chords of a parabola is another parabola.

## UNIT—IV

**7.** *(a)* A variable plane passing through a fixed point $(a, b, c)$ and meets the axes in $A, B, C$ respectively. Show that the locus of point of intersection of plane through $A$, $B$, $C$ and parallel to coordinate plane is $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$.

(b) Prove that the lines $\frac{x}{1}=\frac{y-2}{2}=\frac{z+3}{3}$ and $\frac{x-2}{2}=\frac{y-6}{3}=\frac{z-3}{4}$ are coplanar and find their plane and point of intersection.

**OR**

**8.** (a) Find the length and equation of shortest distance between the lines $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$; $2x-3y+27=0=2y-z+20$.

(b) Find the perpendicular distance of the point $(2, 4, -1)$ from the line $\frac{x+5}{1}=\frac{y+3}{4}=\frac{z-6}{-9}$. Also find its equation.

## UNIT—V

**9.** (a) Prove that the plane $x+2y-z=4$ cuts the sphere $x^2+y^2+z^2-x+z-2=0$ in a circle of radius unity and find the equation of sphere which has this circle for one of its great circle.

(b) Find the equation of the cylinder whose generating lines have the direction cosines $l$, $m$, $n$ and which passes through the circle $x^2+z^2=a^2$, $y=0$.

**OR**

**10.** (a) Find the angle between the lines of intersection of the plane $x-3y+z=0$ and the cone $x^2-5y^2+z^2=0$.

(b) The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the coordinate axes in $A$, $B$, $C$. Prove that the equation of the cone generated by the lines drawn from origin to meet the circle $ABC$ is $yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$.

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