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(CBCS)

(2nd Semester)

MATHEMATICS

SECOND PAPER

(Algebra)

Full Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

(SECTION : A—OBJECTIVE)

(*Marks* : 10)

Tick (✓) the correct answer in the brackets provided : 1×10=10

1. If H and K be two subgroups of a group G such that H has 5 elements and K has 11 elements, then the number of elements of $H \cap K$ is

(a) 1 ( )

(b) 2 ( )

(c) 3 ( )

(d) 4 ( )

2. The number of generators of a cyclic group of order 12 is

(a) 12 ( )

(b) 6 ( )

(c) 4 ( )

(d) 2 ( )

3. When 99^{20} is divided by 25, the remainder is

(a) 79 ( )

(b) 20 ( )

(c) 9 ( )

(d) 1 ( )

4. A homomorphism of a group into itself is called

(a) an automorphism ( )

(b) an endomorphism ( )

(c) an isomorphism ( )

(d) kernel of a homomorphism ( )

5. If $f(x)$ and $g(x)$ be two polynomials of degrees m and n respectively, then $f(x) \cdot g(x)$ is a polynomial of degree

(a) $m \cdot n$ ( )

(b) $m + n$ ( )

(c) m / n ( )

(d) n / m ( )

6. If $f(x)$ is divided by $(ax - b)$, then the remainder is

(a) $f\left(\frac{b}{a}\right)$ ( )

(b) $f\left(-\frac{b}{a}\right)$ ( )

(c) $f(-a)$ ( )

(d) $f(a)$ ( )

7. The equation $x^4 - 3x^2 + 2x + 7 = 0$ has

(a) one positive, one negative and two imaginary roots ( )

(b) one positive and three negative roots ( )

(c) two positive and two negative roots ( )

(d) three positive and one negative roots ( )

8. If 1 is a root of the equation $x^3 - 3x + 2 = 0$, then the multiplicity of 1 is

(a) 1 ( )

(b) 2 ( )

(c) 3 ( )

(d) 0 ( )

9. If α, β, γ, δ be the roots of the biquadratic equation $a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$, then $\Sigma\alpha\beta\gamma$ equals

(a) $-\frac{a_1}{a_0}$ ( )

(b) $\frac{a_2}{a_0}$ ( )

(c) $\frac{a_3}{a_0}$ ( )

(d) $-\frac{a_3}{a_0}$ ( )

10. $(-1 - i)$ in De Moivre's form is

(a) $\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$ ( )

(b) $\sqrt{2}\left(\cos\frac{5\pi}{4} - i\sin\frac{5\pi}{4}\right)$ ( )

(c) $\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ ( )

(d) $\sqrt{2}\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right)$ ( )

(SECTION : B—SHORT ANSWER)

(*Marks* : 15)

Answer the following : 3×5=15

UNIT—I

1. For any two elements a and b of a group G, show that $(ab)^2 = a^2b^2$ if and only if G is abelian.

OR

2. Prove that every proper subgroup of an infinite cyclic group is cyclic.

UNIT—II

3. Prove that every isomorphic image of a cyclic group is cyclic.

OR

4. If f is a homomorphism of a group G into G' and $f(G)$ be the homomorphic image of G into G', then prove that $f(G)$ is a subgroup of G'.

UNIT—III

5. If a polynomial $f(x)$ be divided by a binomial $(x-h)$, then prove that the remainder is $f(h)$.

OR

6. Show that the polynomial $3x^4+15x^2+10$ is irreducible over the field of rational numbers.

UNIT—IV

7. If p, q, r are positive, then show that $x^4+px^2+qx-r=0$ has one positive root, one negative root and two imaginary roots.

OR

8. Show that every equation which is of an even degree and has its last term negative has at least two real roots, one positive and the other negative.

UNIT—V

9. If α, β, γ be the roots of the cubic equation $x^3+px^2+qx+r=0$, then prove that $\sum \frac{1}{\alpha\beta} = -\frac{p}{q}$.

OR

10. Prove that $(\cos 3\theta + i\sin 3\theta)(\cos 2\theta - i\sin 2\theta) = \cos\theta - i\sin\theta$.

(SECTION : C—DESCRIPTIVE)

(*Marks* : 50)

Answer the following : 10×5=50

UNIT—I

1. *(a)* Show that the set of all positive rational numbers forms an abelian group under the composition defined by $a * b = \frac{ab}{2}$. 4

(b) Prove that any two right (left) cosets of a subgroup are either disjoint or identical. 4

(c) Find the order of the permutation $(1\ 3)(2\ 3\ 4)(1\ 2)$ in S_4. 2

OR

2. *(a)* Prove that the set $A = \{1, \omega, \omega^2\}$ forms a group with respect to multiplication composition, where ω is the cube root of unity. 4

(b) Show that the order of a cyclic group is equal to the order of its generator. 4

(c) Prove that the identity element of a group is unique. 2

UNIT—II

3. *(a)* State and prove Lagrange's theorem on the order of a group. 5

(b) Let $f : G \to G'$ be a group homomorphism. Prove that $\ker f = \{e\}$ if and only if f is an isomorphism. 5

OR

4. *(a)* If p is a prime number and a is any integer, then prove that $a^p \equiv a \pmod{p}$. 5

(b) Show that every group of prime order is abelian. 5

UNIT—III

5. *(a)* State and prove division algorithm. 1+5=6

(b) If a polynomial $f(x)$ of degree $n > 2$ is divided by $(x-\alpha)^2$, then prove that the remainder is $(x-\alpha)f'(\alpha)+f(\alpha)$. 4

OR

6. *(a)* Find the remainder when $x^5 - x^4 + 4x^2 + x + 4$ is divided by $(x+1)(x-2)$. 5

(b) Expand $x^5 - 6x^3 + x^2 - 1$ in powers of $x+1$. 5

UNIT—IV

7. *(a)* Find the range of the values of k for which the roots of the equation $x^4 + 4x^3 - 8x^2 + k = 0$ are all real. 5

(b) State Descartes' rule of sign. Hence find the nature of roots of the equation $x^4 - 7x^2 + 1 = 0$. 2+3=5

OR

8. State the fundamental theorem of algebra. Prove that every algebraic equation of nth degree has exactly n roots, real or imaginary. 2+8=10

UNIT—V

9. *(a)* Solve $x^3 - 18x - 35 = 0$ using Cardan's method. 6

(b) Solve the equation $x^8 - 14x^4 + 1 = 0$ by using De Moivre's theorem. 4

OR

10. *(a)* If α, β, γ are the roots of the equation $x^3 + qx + r = 0$, then find the equation whose roots are $\frac{1}{\alpha}+\frac{1}{\beta}, \frac{1}{\beta}+\frac{1}{\gamma}, \frac{1}{\gamma}+\frac{1}{\alpha}$. 6

(b) Remove the term x^2 from the equation $x^3 - 12x^2 - 6x - 10 = 0$. 4

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