MATH260 (MAJOR) **Student's Copy**

2025

(NEP–2020)

(4th Semester)

MATHEMATICS (MAJOR)

(Real Analysis)

Full Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

(SECTION : A—OBJECTIVE)

(*Marks* : 10)

Tick ☑ the correct answer in the boxes provided : 1×10=10

1. Let $S_n = \left[a + \frac{2}{n},\ a+3\right]$, $n \in \mathbb{N}$, $\forall a \in \mathbb{R}$. Then $\bigcup_{n\in\mathbb{N}} S_n$ is

(a) closed but not open ☐

(b) neither closed nor open ☐

(c) open but not closed ☐

(d) both closed and open ☐

2. Let S' denote the derived set of a set S and $\bar{S}$ denote the closure of S, then

(a) $\bar{S}' = S'$ ☐

(b) $\bar{S}' = S$ ☐

(c) $\bar{S}' = \bar{S}$ ☐

(d) $\bar{S}' = \emptyset$ ☐

3. Which of the following sets is an open set?

(a) The set of rational numbers ☐

(b) The set $S = \left\{\frac{1}{n} : n \in \mathbb{N}, \text{ the set of natural numbers}\right\}$ ☐

(c) The closed interval $[a, b]$ ☐

(d) The subset $S = \{(x, y) : x^2 + y^2 < 1,\ x, y \in \mathbb{R}\}$ of $\mathbb{R}^2$ ☐

4. Which of the following functions is continuous at $(0, 0)$?

(a) $f(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$ ☐

(b) $f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$ ☐

(c) $f(x, y) = \begin{cases} \dfrac{x^2 - x\sqrt{y}}{x^2 + y}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$ ☐

(d) $f(x, y) = \begin{cases} \dfrac{x^4 - y^4}{x^4 + y^4}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$ ☐

5. Every continuous function with a closed and bounded domain in $\mathbb{R}^n$ is

(a) bounded and attains its bounds ☐

(b) unbounded and does not necessarily attain its bounds ☐

(c) bounded and does not necessarily attain its bounds ☐

(d) unbounded and attains its bounds ☐

6. Let f be a real-valued function with an open domain in $\mathbb{R}^n$. Then the function admits of directional derivatives at every point where

(a) it is continuous ☐

(b) it possesses continuous first-order partial derivatives ☐

(c) it possesses second-order partial derivatives ☐

(d) it possesses nth-order partial derivatives ☐

7. Let $u = x + y - z,\ v = x - y + z,\ w = x^2 + y^2 + z^2 - 2yz$. Then the value of $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$ is

(a) xy ☐

(b) xyz ☐

(c) 0 ☐

(d) 1 ☐

8. If $f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & x^2 + y^2 \neq 0 \\ 0, & x = y = 0 \end{cases}$, then the value of $f_{xy}(0, 0)$ is

(a) 0 ☐

(b) 1 ☐

(c) −1 ☐

(d) Does not exist ☐

9. For the function $f(x, y) = \begin{cases} \dfrac{x^2y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

(a) the condition of Schwarz's theorem is satisfied ☐

(b) the condition of Young's theorem is satisfied ☐

(c) $f_{xy}(0, 0) = f_{yx}(0, 0)$ ☐

(d) $f_{xy}(0, 0)$ does not exist ☐

10. For the function $f(x, y) = y^2 + 4xy + 3x^2 + x^3$, the point (0, 0) is

(a) a point of maxima ☐

(b) a point of minima ☐

(c) a saddle point ☐

(d) not a critical point ☐

(SECTION : B—SHORT ANSWERS)

(*Marks* : 15)

Answer *five* questions, taking at least *one* from each Unit : 3×5=15

UNIT—I

1. Prove that the interior of a set S is the largest open subset of S.

2. Prove that the derived set of a set is closed.

UNIT—II

3. Prove that the image of a compact set under a continuous function is compact.

4. Let $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^n$. Then prove that f is continuous if and only if for every sequence $\{x_n\}$ in D converging to x_0, $f(x_n)$ also converges to $f(x_0)$.

UNIT—III

5. Find the directional derivative of $f(x, y) = x^2y^3 - y^4$ at the point (2, 1) in the direction of $\theta = \frac{\pi}{4}$.

6. If $u = x + y + z$, $u^2v = y + z$, $u^3w = z$, show that $\frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{1}{u^5}$.

UNIT—IV

7. Prove that the function $f(x, y) = x^2 - 2xy + y^2 + x^4 + y^4$ has a minima at the origin.

8. Using Taylor's theorem, find the expansion of $f(x, y) = \sin 2x + \cos y$ about the origin up to the second degree.

(SECTION : C—DESCRIPTIVE)

(*Marks* : 50)

Answer *five* questions, taking at least *one* from each Unit : 10×5=50

UNIT—I

1. (*a*) State and prove Bolzano-Weierstrass theorem. Show by example that boundedness is not a necessary condition for an infinite set to have a limit point. 6

   (*b*) If a sequence of closed intervals $[a_n, b_n]$ is such that

$$[a_{n+1}, b_{n+1}] \subset [a_n, b_n] \text{ and } \lim_{n \to \infty} (a_n - b_n) = 0$$

   then prove that there is one and only one point common to all the intervals. 4

2. (*a*) State and prove Heine-Borel theorem. 6

   (*b*) Prove that a subset S of $\mathbb{R}^n$ is closed if and only if its complement is open. 4

UNIT—II

3. (*a*) Define uniform continuity of a function in $\mathbb{R}^n$. Prove that a function continuous on a compact set is uniformly continuous. 1+5=6

   (*b*) Show that the function

$$f(x, y) = \begin{cases} \dfrac{2x^4 + 3y^4}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

   is continuous at (0, 0). 4

4. (a) Prove that a function $f : D \to \mathbb{R}$, $D \subset \mathbb{R}^n$ is continuous if and only if $f^{-1}(U)$ is open in D for every open set U in $\mathbb{R}$. 5

(b) Let $f : D \to \mathbb{R}$, $D \subset \mathbb{R}^n$, where D is a convex set. Show that f assumes every value between $f(x)$ and $f(y)$, $\forall x, y \in D$. 5

UNIT—III

5. (a) If the roots of the equation in λ given by

$$(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$$

are u, v, w, then prove that

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = -\frac{2(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$$ 5

(b) If f_x exists throughout a neighbourhood of a point (a, b) in some domain in $\mathbb{R}^2$ and $f_y(a, b)$ exists, then prove that for any point $(a+h, b+k)$ of this neighbourhood

$$f(a+h, b+k) - f(a, b) = hf_x(a+\theta h, b+k) + k\{f_y(a, b) + \eta\}$$

where $0 < \theta < 1$ and η is a function of k which tends to 0 with k. 5

6. (a) Prove that a function which is differentiable at a point possesses the first-order partial derivatives at that point but the converse is not necessarily true. 6

(b) Show that the function

$$f(x, y) = \begin{cases} (x+y)\sin\left(\dfrac{1}{x+y}\right), & x + y \neq 0 \\ 0, & x + y = 0 \end{cases}$$

is continuous at $(0, 0)$ but its first-order partial derivatives do not exist at $(0, 0)$. 4

UNIT—IV

7. *(a)* State and prove Young's theorem. 1+5=6

(b) State Taylor's theorem for two variables. Hence, show that for $0 < \theta < 1$

$$\sin x \sin y = xy - \frac{1}{6}\{(x^3 + 3xy^2)\cos(\theta x)\sin(\theta y) +$$

$$(y^3 + 3x^2y)\sin(\theta x)\cos(\theta y)\}$$ 1+3=4

8. Examine any *two* of the following functions for extreme values : 5×2=10

(i) $f(x, y) = \frac{x^3}{3} + y^2 + 2xy - 6x - 3y + 4$

(ii) $f(x, y) = x^2y^2 - 5x^2 - 8xy - 5y^2$

(iii) $f(x, y) = 4x^2 - xy + 4y^2 + x^3y + xy^3 - 4$

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