2025

(NEP—2020)

(2nd Semester)

MATHEMATICS (MAJOR)

(Elementary Number Theory)

Full Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

(SECTION : A—OBJECTIVE)

(*Marks* : 10)

Tick (✓) the correct answer in the brackets provided : $1 \times 10 = 10$

1. The greatest common divisor of 24531 and 435 is

(*a*) 3 ( )

(*b*) 4 ( )

(*c*) 5 ( )

(*d*) 7 ( )

2. The least common multiple of 3562 and 273 is

(*a*) 74724 ( )

(*b*) 3562 ( )

(*c*) 74725 ( )

(*d*) 76748 ( )

3. A solution of the linear congruence $7x \equiv 5 \pmod 8$ is

(a) $x \equiv 2 \pmod 8$ ( )

(b) $x \equiv 3 \pmod 8$ ( )

(c) $x \equiv 4 \pmod 8$ ( )

(d) $x \equiv 5 \pmod 8$ ( )

4. Which of the following is complete residue system modulo 11?

(a) 0, 2, 12, 3, 15, 19, 21, 9, 7, 16, 17 ( )

(b) 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22 ( )

(c) − 4, − 3, − 2, − 1, 0, 1, 2, 3, 4, 5, 6 ( )

(d) 0, 1, 2, 3, 4, 5, 14, 15, 16, 17, 18 ( )

5. The remainder when 8^{103} is divided by 103 is

(a) 0 ( )

(b) 1 ( )

(c) 103 ( )

(d) 8 ( )

6. Which of the following is true?

(a) For any integer $n > 2$, Euler's function $\phi(n)$ is even ( )

(b) For any integer $n > 2$, Euler's function $\phi(n)$ is odd ( )

(c) For any integer $n > 2$, Euler's function $\phi(n)$ is rational number ( )

(d) For any integer $n > 2$, Euler's function $\phi(n)$ is zero ( )

7. The congruence $x^2 - x + 4 \equiv 0 \pmod 5$ has

(a) no solution ( )

(b) one solution ( )

(c) two solutions ( )

(d) four solutions ( )

8. The value of Legendre symbol of $\left(\frac{29}{53}\right)$ is

(a) -1 ( )

(b) 1 ( )

(c) 0 ( )

(d) 2 ( )

9. Which Fibonacci number is known as the golden ratio, denoted by the Greek letter phi(ϕ)?

(a) 1·317 ( )

(b) 1·318 ( )

(c) 0·617 ( )

(d) 1·618 ( )

10. If $n = 7056$, then the number of positive divisors of n, $\tau(n)$ is

(a) 1 ( )

(b) 0 ( )

(c) 45 ( )

(d) 90 ( )

(SECTION : B—SHORT ANSWERS)

(*Marks* : 15)

Answer *five* questions, taking at least *one* from each Unit : 3×5=15

UNIT—I

1. Prove that if $a \mid b$ and $b \mid c$, then $a \mid c$.

2. Prove that the number of primes is infinite.

UNIT—II

3. If p is a positive prime and n is any positive integer, prove that

$$\phi(1) + \phi(p) + \phi(p^2) + \ldots + \phi(p^{n-1}) + \phi(p^n) = p^n$$

4. Define complete residue system and reduced residue system with example.

UNIT—III

5. Find all odd primes p such that 3 is a quadratic residue modulo p.

6. Let p be an odd prime. Prove that

$$\left(\frac{ab}{p}\right)=\left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$$

UNIT—IV

7. Prove that the Mobius μ function is multiplicative.

8. Prove that any two consecutive terms of Fibonacci sequence are relatively prime.

(SECTION : C—DESCRIPTIVE)

(*Marks* : 50)

Answer *five* questions, taking at least *one* from each Unit : $10\times5=50$

UNIT—I

1. *(a)* State and prove division algorithm. $1+5=6$

   (b) Find the number of distinct positive integral divisors and their sum for the integer 56700. 4

2. *(a)* Let a and b integers, not both zero and let d be a positive integer. Prove that $d=\gcd(a, b)$ iff d satisfies *(i)* $d \mid a$ and $d \mid b$, and *(ii)* $c \mid a$ and $c \mid b$, then $c \mid d$. 6

   (b) For positive integers a and b, prove that $\gcd(a, b)\cdot \text{lcm}(a, b)=ab$. 4

UNIT—II

3. *(a)* Prove that the congruence $ax \equiv b \pmod{m}$ has a solution if and only if the greatest common divisor of a and m divides b, i.e., $(\gcd(a, m) \mid b)$. 6

   (b) Show that $16!+86$ is divisible by 323 by using Wilson's theorem. 4

4. *(a)* State and prove Fermat's theorem. $1+5=6$

   (b) Solve the linear congruence

$$13x \equiv 9 \pmod{25}$$

4

UNIT—III

5. *(a)* Let p be an odd prime and $\gcd(a, p) = 1$. Prove that a is a quadratic residue of p if and only if 6

$$a^{\left(\frac{p-1}{2}\right)} \equiv 1 \pmod{p}$$

(b) Solve $353x \equiv 254 \pmod{400}$. 4

6. *(a)* Using Chinese remainder theorem, solve the following systems of equations : 6

$$x \equiv 3 \pmod{11}$$
$$x \equiv 5 \pmod{19}$$
$$x \equiv 10 \pmod{29}$$

(b) Find all solutions of $x^2 + x + 7 \equiv 0 \pmod{3}$. 4

UNIT—IV

7. *(a)* Let p be a prime and n a positive integer. Prove that the exponent e such that $p^e \mid n!$ is 6

$$\sum_{k=1}^{\infty} \left(\frac{n}{p^k}\right)$$

(b) If f is a multiplicative arithmetic function and F is defined by $F(n) = \Sigma_{d|n} f(d)$, prove that F is also multiplicative. 4

8. *(a)* Let F and f be two arithmetic functions related by the equality

$$f(n) = \sum_{\frac{a}{n}} \mu(a)\, F\left(\frac{n}{a}\right)$$

Prove that $F(n) = \Sigma_{a|n} f(a)$. 6

(b) For each integer $n \geq 1$, prove that 4

$$\Sigma_{d|n}\, \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

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