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(CBCS)

(6th Semester)

MATHEMATICS

NINTH PAPER

(Modern Algebra)

Full Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

(SECTION : A—OBJECTIVE)

(*Marks* : 10)

Put a Tick (✓) mark against the correct answer in the brackets provided : 1×10=10

1. An element a of a group G is self-conjugate if and only if $\forall\ x \in G$

(a) $(ax)^2 = ax$ ( )

(b) $a = x$ ( )

(c) $xa = ax$ ( )

(d) $a = xa$ ( )

2. A subgroup of index 2 is

(a) cyclic ( )

(b) abelian ( )

(c) normal ( )

(d) None of the above ( )

3. Which of the following is *not* an integral domain?

(a) $\mathbb{Z}[i] = \{a + ib : a, b \in \mathbb{Z}\}$, the ring of Gaussian integers ( )

(b) The ring of rational numbers ( )

(c) The finite ring $(\{0, 1, 2, 3, 4\}, +_5, \times_5)$ ( )

(d) The finite ring $(\{0, 1, 2, 3, 4, 5\}, +_6, \times_6)$ ( )

4. The ring of integers is

(a) a subring as well as an ideal of the ring of rational numbers ( )

(b) an ideal but not a subring of the ring of rational numbers ( )

(c) a subring but not an ideal of the ring of rational numbers ( )

(d) neither a subring nor an ideal of the ring of rational numbers ( )

5. In a commutative ring with unity, the associate(s) of the zero element 0 is/are

(a) 0 only ( )

(b) 0 and 1 ( )

(c) 1 and -1 ( )

(d) all the elements of the ring ( )

6. Let a be a unit in a Euclidean ring R. Then

(a) a is not a zero divisor ( )

(b) a is not invertible ( )

(c) a is irreducible ( )

(d) a is a prime element ( )

7. Which of the following sets of vectors is linearly independent?

(a) $\{(2, 1, 4), (6, 3, 12)\}$ ( )

(b) $\{(-1, 2, 1), (3, 0, -1), (-5, 4, 3)\}$ ( )

(c) $\{(1, 2, 1), (3, 1, 5), (3, -4, 7)\}$ ( )

(d) $\{(2, -3, 1), (3, -1, 5), (1, -4, 3)\}$ ( )

8. Let $L(\phi)$ denote the linear span of the null set. Then

(a) $L(\phi) = \phi$ ( )

(b) $L(\phi) = \{0\}$, where 0 is the zero vector ( )

(c) $L(\phi) = 0$, where 0 is the zero vector ( )

(d) $L(\phi)$ is an infinite set ( )

9. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation defined by $T(x, y, z) = (y + z, y - z)$. Then the matrix of T with respect to the ordered basis $\{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$ and $\{(1, 0), (0, 1)\}$ is given by

(a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$ ( )

(b) $\begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ ( )

(c) $\begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}$ ( )

(d) $\begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$ ( )

10. Let T be a linear transformation from a vector space V into a vector space W. Then the nullity of T is

(a) the set of all vectors α in V which are mapped onto the zero vector ( )

(b) the dimension of the null space of T ( )

(c) the set of all vectors α in V which are mapped onto the unit vector ( )

(d) the dimension of the range space of T ( )

(SECTION : B—SHORT ANSWERS)

(*Marks* : 15)

Answer the following questions : 3×5=15

UNIT—1

1. Let G be a group and f be a mapping from G into G defined by $f(x) = x^{-1}$, $\forall\ x \in G$. Show that f is an automorphism if and only if G is Abelian.

OR

2. Prove that a subgroup H of a group G is a normal subgroup of G if and only if each left coset of H in G is a right coset of H in G.

UNIT—2

3. Prove that every field is an integral domain.

OR

4. Show that the set of all 2×2 matrices of the form $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$, for all integers a, b is a left ideal but not a right ideal of the ring R of all 2×2 matrices with elements as integers.

UNIT—3

5. Prove that every field is a Euclidean ring.

OR

6. In the quadratic integer ring $\mathbb{Z}[i\sqrt{5}] = \{a + ib\sqrt{5} : a,\ b \in \mathbb{Z}\}$, show that 3 is irreducible but not prime.

UNIT—4

7. Show that if two vectors are linearly independent, then one of them is a scalar multiple of the other.

OR

8. Show that the subset of the vector space $\mathbb{R}^n$ given by $S = \{(a_1, a_2, \cdots, a_n) : a_2 + 2a_3 = 0\}$ is a subspace of $\mathbb{R}^n$.

UNIT—5

9. Show that the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (\sin x, y)$ is not a linear transformation.

OR

10. Find the kernel of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (x + y, y - z)$.

(SECTION : C—DESCRIPTIVE)

(*Marks* : 50)

Answer the following questions : 10×5=50

UNIT—1

1. *(a)* If G is a group and H is any subgroup of G, and if N is any normal subgroup of G, then prove that

$$\frac{HN}{N} \cong \frac{H}{H \cap N}$$ 6

(b) If the order of a group G is p^2, where p is prime, then show that G is abelian. 4

OR

2. Prove that the set $I(G)$ of all inner automorphisms of a group G is a normal subgroup of the group of all automorphisms of G. Also prove that $I(G)$ is isomorphic to the quotient group G/Z of G, where Z is the centre of G. 4+6=10

UNIT—2

3. *(a)* Prove that every finite integral domain is a field. 5

(b) Show that the ring of integers is a principal ideal domain. 5

OR

4. *(a)* Show that a commutative ring with unity is a field if it has no proper ideal. 4

(b) If R is a commutative ring with unity, then prove that an ideal S of R is maximal if and only if the factor ring R/S is a field. 6

UNIT—3

5. *(a)* Let f be a homomorphism of a ring R into a ring R'. Show that the kernel of f is an ideal of R. 4

(b) Prove that every non-zero element in a Euclidean ring R is either a unit in R or can be written as a product of a finite number of prime elements of R. 6

OR

6. *(a)* Let D be an integral domain with unity element 1. Show that two non-zero elements $a, b \in D$ are associates if and only if $a \mid b$ and $b \mid a$. 4

(b) Let a and b be two non-zero elements of a Euclidean ring R. Then show that *(i)* if b is a unit in R, then $d(ab) = d(a)$ and *(ii)* if b is not a unit in R, then $d(ab) > d(a)$. 6

UNIT—4

7. *(a)* Prove that every linearly independent subset of a finitely generated vector space $V(F)$ is either a basis or can be extended to form a basis of V. 5

(b) Prove that the union of two subspaces is a subspace if and only if one is contained in the other. 5

OR

8. *(a)* If W is a subspace of a finite-dimensional vector space V, then prove that $\dim(V/W) = \dim V - \dim W$. 6

(b) Show that a superset of a linearly independent set of vectors is linearly independent. 4

UNIT—5

9. *(a)* Let V be a finite-dimensional vector space over the field F and let $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be an ordered basis for V. Let W be vector space over the same field and let $\beta_1, \beta_2, \cdots, \beta_n$ be any n vectors in W. Then prove that there exists a unique linear transformation T from V into W such that $T(\alpha_i) = \beta_i$, $i = 1, 2, \cdots, n$. 5

(b) Let V and W be vector spaces over the same field F, and let T be a linear transformation from V into W. If V is finite-dimensional, then prove that rank T + nullity T = dim V. 5

OR

10. *(a)* Let V be an n-dimensional vector space over the field F, and W be an m-dimensional vector space over the same field F. Then prove that the vector space $L(V, W)$ of linear transformations from V into W is also finite-dimensional and is of dimension mn. 5

(b) Let $T : R^4 \to R^3$ be a linear transformation defined by

$$T(x, y, z, t) = (x - y + z + t,\ x + 2z - t,\ x + y + 3z - 3t)$$

Determine the rank and nullity of T. 5

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