Student's Copy

2025

(CBCS)

(6th Semester)

MATHEMATICS

TENTH PAPER

(Advanced Calculus)

Full Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

(SECTION : A—OBJECTIVE)

(*Marks* : 10)

Tick ☑ the correct answer in the box provided : 1×10=10

1. If $f(x) = x$ and let $P = \{0, 1, 2, 3\}$ be the partition of $[0, 3]$, then the value of $U(P, f)$ is

   (a) $\frac{1}{3}$ ☐

   (b) $\frac{1}{2}$ ☐

   (c) 3 ☐

   (d) 6 ☐

2. If a bounded function f is integrable on $[a, b]$, then

(a) $\lim_{\mu(P)\to\infty} S(P, f) = \int_a^b f\, dx$ ☐

(b) $\lim_{\mu(P)\to 0} S(P, f) = \int_a^b f\, dx$ ☐

(c) $\int_{\underline{a}}^b f\, dx \neq \int_a^{\bar{b}} f\, dx$ ☐

(d) $L(P, f) = U(P, f) = S(P, f)$ ☐

3. If f and g be two positive functions on $[a, b]$ such that $\lim_{x\to a^+} \frac{f(x)}{g(x)} = l$, a non-zero finite number, then

(a) $\int_a^b g\, dx$ converges if $\int_a^b f\, dx$ diverges ☐

(b) $\int_a^b f\, dx$ converges if $\int_a^b g\, dx$ diverges ☐

(c) $\int_a^b f\, dx$ diverges if $\int_a^b g\, dx$ converges ☐

(d) $\int_a^b f\, dx$ and $\int_a^b g\, dx$ behave alike ☐

4. The improper integral $\int_a^\infty \frac{1}{x^n} dx$ converges if and only if

(a) $n > 1$ ☐

(b) $n < 1$ ☐

(c) $n = 1$ ☐

(d) $n \leq 1$ ☐

5. Let $\phi(y) = \int_a^b f(x, y)\, dx$ is continuous and f_y also exists and continuous in $[a, b; c, d]$. Then ϕ is

(a) continuous ☐

(b) integrable ☐

(c) derivable ☐

(d) None of the above ☐

6. If $\int_0^\infty \frac{\cos mx}{a^2 + x^2} dx = \frac{\pi}{2a} e^{-ma}$, then the value of $\int_0^\infty \frac{x \sin mx}{1 + x^2} dx$ is

(a) $-\frac{\pi}{2} e^{-m}$ ☐

(b) $\frac{\pi}{2} e^{m}$ ☐

(c) $-\frac{\pi}{2} e^{m}$ ☐

(d) $\frac{\pi}{2} e^{-m}$ ☐

7. The value of $\int_C (x^2 dx + xy\, dy)$ taken along the line segment from (1, 0) to (0, 1) is

(a) $\frac{1}{6}$ ☐

(b) $-\frac{1}{2}$ ☐

(c) $\frac{1}{2}$ ☐

(d) $\frac{\pi}{2}$ ☐

8. The value of the integral $\int_{y=1}^{2} \int_{x=0}^{3} (1 + 8xy)\, dx\, dy$ is

(a) 54 ☐

(b) 57 ☐

(c) – 36 ☐

(d) 45 ☐

9. Regarding convergence of sequences in $[a, b]$, which of the following is true?

(a) Every pointwise convergence is uniform convergence ☐

(b) Every uniform convergence is pointwise convergence ☐

(c) Every uniform limit need not be a pointwise limit ☐

(d) All of the above ☐

10. The sequence $\{f_n\}$ of continuous function is uniformly convergent to a function f on $[a, b]$. Then f is also

(a) uniformly convergent ☐

(b) integrable ☐

(c) continuous ☐

(d) differentiable ☐

(SECTION : B—SHORT ANSWERS)

(*Marks* : 15)

Answer the following : 3×5=15

UNIT—I

1. Let f be a bounded function defined on $[a, b]$ and let P be the partition of $[a, b]$. If P^* is a refinement of P, then prove that $U(P, f) \geq U(P^*, f)$.

OR

2. Test the Riemann integrability for the function $f(x) = x$ by dividing $[0, 1]$ into 6 equal intervals.

UNIT—II

3. Test the convergence of the improper integral $\int_0^\infty x^3 e^{-x^2} dx$.

OR

4. Using Frullani's integral, evaluate $\int_0^\infty \frac{\tan^{-1}(ax) - \tan^{-1}(bx)}{x} dx$.

UNIT—III

5. If $f(x, y) = \frac{y^2}{x^2 + y^2}$ and $g(y) = \int_0^1 f(x, y)\, dx$, then evaluate the value of $\int_0^1 f_y(x, 0)\, dx$.

OR

6. Write the statement of Weierstrass' M-test for uniform convergence of improper integral of the type $\int_a^b f(x, y)\, dx$.

UNIT—IV

7. Evaluate the value of $\iint_A xy\,dx\,dy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$.

OR

8. Evaluate the double integral $\int_1^4 \int_0^{\sqrt{y}} e^{x/\sqrt{y}}\,dx\,dy$.

UNIT—V

9. Show, by M_n-test, the sequence of function $f_n(x) = \dfrac{x}{1 + b^2x^2}$ is uniformly convergent in $[k, 1]$ for $k > 0$, but not uniformly convergent in $[0, 1]$.

OR

10. Prove that the sequence $f_n(x) = x^n$ is pointwise convergent on $[0, 1]$ and evaluate the pointwise limit.

(SECTION : C—DESCRIPTIVE)

(*Marks* : 50)

Answer the following : 10×5=50

UNIT—I

1. *(a)* If a function f is bounded and R-integrable on $[a, b]$ and there exists a function F such that $F' = f$ on $[a, b]$, then prove that

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

5

(b) Prove that the function defined by

$$f(x) = \frac{1}{2^n}, \text{ when } \frac{1}{2^{n+1}} < x \le \frac{1}{2^n}, \text{ for } n = 0, 1, 2, \ldots$$

and $f(0) = 0$ is R-integrable and show that $\int_0^1 f\,dx = \dfrac{2}{3}$.

3+2=5

OR

2. *(a)* If $\int_a^b f\,dx$ and $\int_a^b g\,dx$ both exist and $g(x)$ keeps the same sign over $[a, b]$, then prove that there exists a number μ lying between the bounds of f such that

$$\int_a^b f(x)\,g(x)\,dx = \mu \int_a^b g\,dx$$ 5

(b) If a function f is continuous on $[0, 1]$, then find the value of

$$\lim_{n\to\infty} \int_0^1 \frac{nf(x)}{1+n^2x^2}\,dx$$ 5

UNIT—II

3. *(a)* Prove that the improper integral $\int_a^b f(x)\,dx$ is convergent at a if and only if to every $\varepsilon > 0$ there exists a corresponding $\delta > 0$ such that

$$\left| \int_{a+\lambda_1}^{a+\lambda_2} f(x)\,dx \right| < \varepsilon \text{ for } 0 < \lambda_1, \lambda_2 < \delta$$ 5

(b) Prove that every absolutely convergent integral is convergent. 5

OR

4. *(a)* If $\phi(x)$ is bounded and monotonic in $[a, \infty)$ and $\int_a^\infty f(x)\,dx$ is convergent at ∞, then prove that $\int_a^\infty f(x)\,\phi(x)\,dx$ is convergent at ∞. 5

(b) By Cauchy's test, show that $\int_0^\infty \frac{\sin x}{x}\,dx$ is convergent. 5

UNIT—III

5. *(a)* If $|a| \le 1$, then show that

$$\int_0^\pi \log(1 + a\cos x)\,dx = \pi \log\left(\frac{1}{2} + \frac{1}{2}\sqrt{1-a^2}\right)$$ 5

(b) Evaluate the value of the improper integral

$$\int_0^\infty e^{-x^2} \cos \alpha x\,dx$$ 5

OR

6. (a) Prove that uniformly convergent improper integral of a continuous function is itself continuous. 5

(b) Evaluate the value of the improper integral

$$f(y) = \int_0^\infty \frac{\cos yx}{1+x^2}\,dx$$ 5

UNIT—IV

7. (a) Evaluate the integral $\int_0^3 \int_{x^2}^9 x^3 e^{y^3}\,dy\,dx$ by reversing the order of integration with a rough figure. 5

(b) Show that

$$\int_0^1 \left\{ \int_0^1 \frac{x^2-y^2}{x^2+y^2}\,dy \right\} dx = \int_0^1 \left\{ \int_0^1 \frac{x^2-y^2}{x^2+y^2}\,dx \right\} dy$$

i.e., change in order of integration is permissible. 5

OR

8. (a) Change the order of integration in the integral $\int_0^1 \int_{x^2}^{2-x} xy\,dx\,dy$ and hence evaluate it with a rough figure. 5

(b) Prove that

$$\int_0^1 \left\{ \int_0^1 \frac{(x-y)}{(x+y)^3}\,dy \right\} dx = \frac{1}{2}$$

but the value changes its sign as the order of integration interchanges. 5

UNIT—V

9. (a) Show that $f_n(x) = x^n$ is uniformly convergent on $[0, k]$, where k is a number less than 1 and only pointwise convergent in $[0, 1]$. 5

(b) Show that a series of function Σf_n will converge uniformly (and absolutely) on $[a, b]$ if there exists a convergent series ΣM_n of positive numbers such that $\forall\, x \in [a, b] \,|f_n(x)| \le M_n \;\forall\, n$. 5

OR

10. *(a)* If a sequence $\{f_n\}$ converges uniformly to f on $x \in [a, b]$ and f_n be integrable $\forall\, n$, then prove that f is integrable and

$$\int_a^x f(x)\, dx = \lim_{n \to \infty} \int_a^x f_n(x)\, dx$$ 5

(b) Examine whether the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^3(1+n)x^2}$$

can be differentiated term-by-term. 5

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