2024

(NEP–2020)

(3rd Semester)

MATHEMATICS (MAJOR)

(Modern Algebra—I)

Full Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

(SECTION : A—OBJECTIVE)

(*Marks* : 10)

Tick ☑ the correct answer in the boxes provided : 1×10=10

1. The inverse of the element a in the group of all positive rational numbers under the composition $a * b = \frac{ab}{2}$ is

   (a) 2 ☐

   (b) $a / 4$ ☐

   (c) $4 / a$ ☐

   (d) $4a$ ☐

2. The number of binary compositions on a finite set A having n elements is

(a) 2^{n^2} ☐

(b) n^{n^2} ☐

(c) n^n ☐

(d) n^2 ☐

3. In $(Z_6, \oplus_6)$, where $Z_6 = \{0, 1, 2, 3, 4, 5\}$ and $\oplus_6$ is the addition modulo 6, the order of 4 is

(a) 2 ☐

(b) 3 ☐

(c) 4 ☐

(d) 5 ☐

4. If a and b are any two elements of a group G and H is any subgroup of G, then $Ha = Hb$ if and only if

(a) $ab \in H$ ☐

(b) $a^{-1}b \in H$ ☐

(c) $ab^{-1} \in H$ ☐

(d) $(ab)^{-1} \in H$ ☐

5. Let $T = \{\pm 1, \pm i, \pm j, \pm k\}$ be a quaternion group. Then

(a) T is non-Abelian with order 4 ☐

(b) T is non-Abelian with order 8 ☐

(c) T is Abelian with order 4 ☐

(d) T is Abelian with order 8 ☐

6. Which of the following is ***incorrect***?

(a) A group of order 1 is Abelian. ☐

(b) Every commutative group is Abelian. ☐

(c) In a group $(G, \times)$, where $\times$ denotes ordinary multiplication, a^{-1} and b are commute if a and b are commute $\forall a, b \in G$. ☐

(d) If every element of a group G is its own inverse, then G is non-Abelian. ☐

7. The order of a finite Abelian group $(G, \oplus_4)$, where $G = \{0, 1, 2, 3\}$ and $\oplus_4$ as addition modulo 4, is

(a) 6 ☐

(b) 5 ☐

(c) 4 ☐

(d) 3 ☐

8. Let a be any element of an Abelian group G such that $ax = xa \; \forall x \in G$. Then

(a) $N(a) > G$ ☐

(b) $N(a) < G$ ☐

(c) $N(a) = G$ ☐

(d) $N(a) = \{ \}$ ☐

where $N(a)$ is normalizer of an element of G.

9. The generator of the cyclic group $(\omega, \omega^2, 1)$ is

(a) ω and ω^2 ☐

(b) ω only ☐

(c) ω^2 only ☐

(d) ω, ω^2 and ω^3 ☐

10. The inverse of the product of (1 2) and (2 3) of the symmetric group S_3 is

(a) (1 2 3) ☐

(b) (1) ☐

(c) (1 3) ☐

(d) (1 3 2) ☐

(SECTION : B—SHORT ANSWERS)

(*Marks* : 15)

Answer *five* questions, taking at least *one* from each Unit : 3×5=15

UNIT—I

1. In the set of real numbers, show that the composition $x * y = x^y$ for all $x, y \in \mathbb{R}$ is not associative.

2. Prove that the inverse of the product of two elements of a group G is the product of the inverse taken in the reverse order.

UNIT—II

3. Prove that the intersection of two subgroups of a group G is a subgroup of G.

4. Prove that any two right cosets of a subgroup are either disjoint or identical.

UNIT—III

5. Prove that a group G is an Abelian if $a^2 = e \; \forall a \in G$, where e is the identity element of G.

6. Show that the symmetric group S_n is Abelian for $n = 2$ and non-Abelian other than $n > 2$.

UNIT—IV

7. Find the remainder when 45^{31} is divided by 33.

8. If a finite group of order n contains an element of order n, then prove that the group must be cyclic.

(SECTION : C—DESCRIPTIVE)

(*Marks* : 50)

Answer *five* questions, taking at least *one* from each Unit : $10 \times 5 = 50$

UNIT—I

1. *(a)* Let A be the set of fourth root of unity, i.e., $A = \{1, -1, i, -i\}$. Then show that the algebraic structure $(A, \times)$ is a group, where $\times$ is an ordinary multiplication. 5

(b) Prove that the set $G = \{1, 2, 3, 4, 5, 6\}$ is a group with respect to multiplication modulo 7, using group table. 5

2. *(a)* Prove that the identity element in a group is unique. Also prove that the inverse of each element of a group is unique. 5

(b) If a and b are any two elements of a group G, then show that the equations $ax = b$ and $ya = b$ have unique solutions in G. 5

UNIT—II

3. *(a)* Prove that the necessary and sufficient condition for a non-empty subset K of a group G to be a subgroup is that $r \in K$, $s \in K \Rightarrow rs^{-1} \in K$; where s^{-1} is the inverse of s in G. 5

(b) If H and K are two subgroups of a group G, then show that HK is a subgroup of G if and only if $HK = KH$. 5

4. *(a)* Prove that the order of every element of a finite group is finite and is less than or equal to the order of the group. 5

(b) State and prove Lagrange's theorem on the order of a group. 5

UNIT—III

5. Prove that—

   (a) the set of all nth of unity forms a finite Abelian group of order n with respect to multiplication;

   (b) G is Abelian if it is a group with $(ab)^m = a^m b^m$ for three consecutive integers $m \, \forall a, b \in G$. 7+3=10

6. (a) Show that if G is an Abelian group, then for any integer n such that

$$(ab)^n = a^n b^n \, \forall a, b \in G$$

5

   (b) Prove that a group G is Abelian if and only if

$$(pq)^{-1} = p^{-1} q^{-1} \, \forall p, q \in G$$

5

UNIT—IV

7. (a) Show that every finite group is isomorphic to a permutation group. 7

   (b) Find the order of permutation of (23)(134)(13) in the symmetric group S_4. 3

8. Prove that—

   (a) for any set S, $A(S)$ is a group with respect to composition mapping where $A(S)$ denotes the set of all one to one mapping of S onto itself;

   (b) the normalizer of a group G is a subgroup of G. 5+5=10

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Student's Copy

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