2024

(NEP–2020)

(2nd Semester)

MATHEMATICS (MAJOR)

(Elementary Number Theory)

Full Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

(SECTION : A—OBJECTIVE)

(*Marks* : 10)

Tick (✓) the correct answer in the brackets provided : $1\times10=10$

1. The greatest common divisor of 42823 and 6409 is

(a) 27 ( )

(b) 17 ( )

(c) 13 ( )

(d) 23 ( )

2. The least common multiple of 482 and 1687 is

(a) 3476 ( )

(b) 4786 ( )

(c) 3374 ( )

(d) 5348 ( )

3. If $n = 50000$, then the value of $\tau(n)$ is

(a) 30 ( )

(b) 40 ( )

(c) 50 ( )

(d) 45 ( )

4. Which of the following sets is the complete residue system modulo 3?

(a) $\{-5, 14, 10\}$ ( )

(b) $\{-12, 4, 5\}$ ( )

(c) $\{12, 4, 6\}$ ( )

(d) $\{13, 15, 17\}$ ( )

5. $\varphi(35)$ equals (where φ is an Euler φ-function)

(a) 34 ( )

(b) 42 ( )

(c) 20 ( )

(d) 24 ( )

6. When 7^{23} is divided by 8, the remainder is

(a) 14 ( )

(b) 8 ( )

(c) 7 ( )

(d) 9 ( )

7. If n is a positive integer and a is any integer relatively prime to n, then

(a) $a^{\phi(n)} \equiv 0 \pmod{n}$ ( )

(b) $a^{\phi(a)} \equiv 1 \pmod{a}$ ( )

(c) $a^{\phi(n)} \equiv 1 \pmod{n}$ ( )

(d) $a^{\phi(n)} \equiv 2 \pmod{n}$ ( )

8. If $a = 2$, $p = 13$, then the value of $\left(\frac{a}{p}\right)$ is

(a) -1 ( )

(b) 0 ( )

(c) 1 ( )

(d) 2 ( )

9. If $n = 42$, then the value of $\mu(n)$ is

(a) 0 ( )

(b) -1 ( )

(c) -2 ( )

(d) 1 ( )

10. If $n = 99$, then the value of $\sigma(n)$ is

(a) 445 ( )

(b) 346 ( )

(c) 235 ( )

(d) 156 ( )

(SECTION : B—SHORT ANSWERS)

(*Marks* : 15)

Answer *five* questions, taking at least *one* from each Unit : 3×5=15

UNIT—I

1. If $a \mid bc$ and $\gcd(a, b) = 1$, then prove that $a \mid c$.

2. Prove that $n(n+1)(2n+1)$ is a multiple of 6 for every natural number n.

UNIT—II

3. Solve the linear congruence $13x \equiv 10 \pmod{28}$.

4. Prove that $n^5 - n$ is divisible by 30.

UNIT—III

5. Let p be an odd prime. Prove that

$$\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$$

6. Solve the quadratic congruence $x^2 + 7x + 10 \equiv 0 \pmod{11}$.

UNIT—IV

7. For any positive integer n, prove that

$$f_{n+3} + f_n = 2f_{n+2}$$

8. Prove that $\dfrac{\phi(n)}{n} = \sum_{d \mid n} \dfrac{\mu(d)}{d}$.

(SECTION : C—DESCRIPTIVE)

(*Marks* : 50)

Answer *five* questions, taking at least *one* from each Unit : 10×5=50

UNIT—I

1. (a) State and prove division algorithm for integers. 6

   (b) Find the values of x and y using the Euclidean algorithm satisfying the condition that $\gcd(1769, 2378) = 1769x + 2378y$. 4

2. (a) Prove the fundamental theorem of arithmetic. 5

   (b) Find the number of distinct positive integral divisors and their sum for the integer 55000. 5

UNIT—II

3. (a) State and prove Fermat's theorem. 5

   (b) If $a \equiv b \pmod{m_1}$, $a \equiv b \pmod{m_2}$ and m is the least common multiple of m_1 and m_2, then prove that $a \equiv b \pmod{m}$. Also prove the converse. 5

4. (a) Show that the set of $\{-3, -1, 3, 14, 12, 37, 57\}$ is the complete residue system modulo 7. 5

   (b) State and prove Euler's theorem. 5

UNIT—III

5. (a) Using Chinese remainder theorem, find the least positive integer x which satisfies

$$x \equiv 5 \pmod{7}$$
$$x \equiv 7 \pmod{11}$$
$$x \equiv 3 \pmod{13}$$ 5

(b) Let p be an odd prime and $\gcd(a, p) = 1$. Prove that a is a quadratic residue of p if and only if $a^{(p-1)/2} \equiv 1 \pmod{p}$. 5

6. (a) Find the solutions of polynomial congruence

$$f(x) = x^2 + x + 7 \equiv 0 \pmod{3}$$ 5

(b) Let p be an odd prime and $\gcd(a, p) = 1$. Prove that

$$\left(\frac{a}{p}\right) = (-1)^n$$

where n denotes the number of integers in the set $S = \left\{a, 2a, 3a, \ldots, \left(\frac{p-1}{2}\right)a\right\}$, where remainders upon division by p exceed $\frac{p}{2}$. 5

UNIT—IV

7. (a) Prove that the functions ϕ, μ, σ, τ are all multiplicative arithmetic functions. 5

(b) Prove that for each integer $n \geq 1$,

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1 \end{cases}.$$ 5

8. *(a)* Prove that the greatest common divisor of two Fibonacci numbers is again a Fibonacci number, specially, $\gcd(u_m, u_n) = u_d$, where $d = \gcd(m, n)$. 5

(b) If f is a multiplicative arithmetic function and F is defined by $F(n) = \sum_{d|n} f(d)$, then prove that F is also multiplicative. 5

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