MATH/VI/CC/10

Student's Copy

2024

(CBCS)

(6th Semester)

MATHEMATICS

TENTH PAPER

(Advanced Calculus)

Full Marks : 75 Time : 3 hours

The figures in the margin indicate full marks for the questions

(SECTION : A-OBJECTIVE)

(Marks: 10)

Put a Tick I mark against the correct answer in the boxes provided : 1×10=10

- 1. If f(x) = x over [0, 1] and $P = \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, \frac{n}{n} = 1\right\}$ be the partition, then the value of the lower Riemann integral $\int_{0}^{1} f dx$ is
 - (a) $\frac{1}{3}$ \Box (b) $\frac{1}{2}$ \Box (c) $-\frac{1}{2}$ \Box (d) None of the above \Box

2. For any two partitions P_1 , P_2 of the domain of a bounded function f

(a) $L(P_1, f) \le U(P_2, f)$ (b) $L(P_1, f) < U(P_2, f)$ (c) $L(P_2, f) \le U(P_1, f)$ (d) $U(P_2, f) \le L(P_1, f)$ (e) $L(P_1, f) \le L(P_1, f)$

/543

| Contd.

- 3. If f and g be two positive functions such that f(x) ≤ g (x) ∀x ∈ [a, b], then
 (a) ∫_a^bg dx converges if ∫_a^bf dx converges □
 (b) ∫_a^bf dx converges if ∫_a^bg dx converges □
 (c) ∫_a^bf dx diverges if ∫_a^bg dx diverges □
 (d) None of the above □
- 4. The value of the improper integral
 - $\int_{0}^{\infty} \frac{e^{-ax} e^{-bx}}{x} dx$ (a) is $\frac{\pi}{2} \log\left\{\frac{a}{b}\right\}$ \Box (b) is $\log\left\{\frac{a}{b}\right\}$ \Box (c) is $\log\left\{\frac{b}{a}\right\}$ \Box (d) does not exist \Box
 - 5. The value of the improper integral $\int_0^\infty e^{-x^2} \cos \alpha x \, dx$ is equal to
 - (a) $\frac{\sqrt{\pi}}{2}$ (b) $\frac{\sqrt{\pi}}{2}e^{-\frac{\alpha}{4}}$ (c) $\frac{\pi}{\sqrt{2}}e^{-\frac{\alpha^2}{4}}$ (d) $\frac{\sqrt{\pi}}{2}e^{-\frac{\alpha^2}{4}}$
 - 6. The value of the improper integral $\int_0^{\pi} \frac{dx}{a + b \cos x}$, if a is positive and |b| < a is

(a)
$$\frac{2\pi}{(a^2 - b^2)^{1/2}}$$
 (b) $\frac{2\pi}{(a^2 - b^2)^{3/2}}$ (c) $\frac{\pi}{(a^2 - b^2)^{1/2}}$ (d) $\frac{\pi}{(a^2 - b^2)^{3/2}}$ (e)

/543

| Contd.

- 7. The value of the double integral $\int_{1}^{1} \int_{1}^{1} \frac{x-y}{x+y} dxdy$ is
 - (a) 0 (b) $-\frac{1}{2}$ (c) $\frac{1}{2}$ (d) $\frac{\pi}{2}$ (d) $\frac{\pi$
- 8. The value of the integral $\int_C \frac{dx}{x+y}$, where C is the curve $x = at^2$, y = 2at, $0 \le t \le 2$, is (a) $\frac{1}{6}$ \Box (b) $\log 4$ \Box (c) $-\frac{1}{6}$ \Box (d) $\log 2$ \Box
- By M_n-test, the sequence {f_n} converges uniformly to f on [a, b] if and only if
 - (a) $M_n \to 0 \text{ as } n \to 0$
 - (b) $M_n \to \infty \text{ as } n \to 0$
 - (c) $M_n \to 0 \text{ as } n \to \infty$
 - (d) None of the above

10. The sequence of function given by $f_n(x) = \frac{n}{x+n}$ is

(a) uniformly convergent in [0, k], whatever k may be

(b) only pointwise convergent in [0, k], whatever k may be

(c) non-convergent at all in [0, k], whatever k may be

(d) uniformly convergent in [0, ∞)

		3×5=15		is R-integrable over				while f is not						and only if to	$0 < \lambda_1, \lambda_2 < \delta$.		al		l Contid	
(SECTION : B-SHORT ANSWERS)	(Marks : 15)	Answer the following questions :	UNITI	1. If a function f is monotonic on $[a, b]$, then prove that it is R-integrable over $[a, b]$.	OR	2. Let f be a real-valued function defined by	$f(x) = \begin{cases} 1, \text{ when } x \text{ is rational number} \\ -1, \text{ when } x \text{ is irrational number} \end{cases}$	Prove that $ f $ is R-integrable on any interval $[a, b]$ while f is not R-integrable.	UNIT	3. Using Dirichlet's test, prove that	$\int_0^\infty \frac{\cos x}{x} dx$	is convergent at ∞.	OR	4 . Prove that the improper integral $\int_a^b f dx$ converges at a if and only if to	every $\varepsilon > 0$, there corresponds $\delta > 0$ such that $\int_{\alpha+\lambda_1} f dx < \varepsilon, 0 < \lambda_1, \lambda_2 < \delta$.	UNITIII	Establish the uniform convergence of the improper integral	$\int_0^\infty \frac{y}{x^2 + y^2} dx, \ (0 < c \le y \le d)$	4	
		Answer the		1. If a fut $[a, b]$.		2. Let f		Prove R-int		3. Using		is col		4. Prove	cvery		5. Estat		/543	

OK	6. Given that $\int_0^\infty \frac{\cos mx}{a^2 + x^2} dx = \frac{\pi}{2a} e^{-ma}$. Prove that $\int_0^\infty \frac{\sin mx}{x(1 + x^2)} dx = \frac{\pi}{2} (1 - e^{-m})$.	UNITIV	7. Show that	$\int_0^{\pi} \int_0^{\sin x} y dy dx = \frac{\pi}{2}$	OR	8. Show that $\int_C [(x-y)^3 dx + (x-y)^3 dy] = 3\pi a^4$, taken along the circle $x^2 + y^2 = a^2$ in the counter-clockwise sense.	UNITV	9. Prove that the sequence $f_n(x) = x^n$ is pointwise convergent on [0, 1] and evaluate the pointwise limit. OR	10. Show that $f_n(x) = \frac{1}{x+n}$ is uniformly convergent in any interval [0, b], where $b > 0$.	(SECTION : C-DESCRIPTIVE)	(Marks : 50)	Answer the following questions : 10×5=50	UNITI	1. (a) Prove that a necessary and sufficient condition for the integrability of bounded function f is that to every $\varepsilon > 0$, there exists a corresponding $\delta > 0$ such that $U(P, f) - L(P, f) < \varepsilon$ for every partition P of $[a, b]$ with norm $\mu(P) < \delta$.
----	---	--------	--------------	--	----	---	-------	---	---	-----------------------------	----------------	--	-------	---

0R

/543

Contd.

21/2+21/2=5 S 5 **2.** (a) If $\int_{a}^{b} f dx$ and $\int_{a}^{b} g dx$ both exist and g(x) keeps the same sign over [a, b], then prove that there exists a number μ lying between the (b) Compute the value of $\int_{-1}^{1} f dx$, where f(x) = |x| by dividing the interval converges absolutely for p > 1 but only conditionally for 0 .3. (a) Examine the convergence of the following functions : Making use of the Riemann sum S(P, f), show that $\int_{a}^{b} f(x)g(x) \, dx = \mu \int_{a}^{b} g \, dx$ $\int_1^2 f(x)dx = \frac{11}{2}$ $\int_{1}^{\infty} \frac{\sin x}{x^{p}} dx$ UNIT--II [-1, 1] into 2n equal sub-intervals. is convergent if and only if n > 0. ы В В g bounds of f such that 4. (a) Prove that the integral (ii) $\int_0^\infty \frac{x \tan^{-1} x}{(1+x^4)^{1/3}} dx$ $\int_{1}^{\infty} x^{n-1} e^{-x} dx$ where f(x) = 3x + 1. (i) $\int_{2}^{\infty} \frac{2x^2}{x^4 - 1} dx$ Show that (q) (q)

S

ເດ

ŝ

/543

ø

[Contd.

(b) If
$$\phi(x)$$
 is bounded and monotonic in $[\alpha, \infty)$ and $\int_{\alpha}^{\infty} f(x) dx$ is convergent at

 ∞ , then prove that $\int_a^\infty f(x)\phi(x)dx$ is convergent at ∞ .

S

UNIT-III

- S Prove that uniformly convergent improper integral of a continuous function is itself continuous. 5. (a)
 - Show that (q)

$$\int_0^{\pi/2} \log(a^2 \cos^2 \theta + b^2 \sin^2 \theta) d\theta = \pi \log\left(\frac{a+b}{2}\right)$$

g

6. (a) Let $\phi(y) = \int_{a}^{b} f(x, y) dx$, is continuous and f_{y} also exists and continuous

S in [a, b; c, d]. Prove that ϕ is derivable and $\phi'(y) = \int_a^b f_y(x, y) dx \quad \forall y \in [c, d].$

(b) Prove that the function

$$p(y) = \int_{\alpha}^{b} f(x, y) \, dx$$

is continuous in [c, d].

ŝ

7. (a) Evaluate the integral

$$\int_{0}^{8} \int_{y}^{2} \sqrt{x^{4} + 1} dx dy$$

by reversing the order of integration with a rough figure. Show that if 0 < h < 1(q)

ß

1[1 1 [1

$$\int_{n}^{1} \left\{ \int_{n}^{1} f(x, y) dx \right\} dy = \int_{n}^{1} \left\{ \int_{n}^{1} f(x, y) dy \right\} dx = 0$$

$$\lim_{n \to 0} \int_{0}^{1} \int_{0}^{1} f(x, y) dx \right\} dy \neq \int_{0}^{1} \left\{ \int_{0}^{1} f(x, y) dy \right\} dx, \text{ where } f(x, y) = \frac{y^2 - x^2}{(y^2 + x^2)^2}.$$

1543

Contd.

g

Change the order of integration in the integral 8. (a)

$$\int_{0}^{1} dx \int_{0}^{\sqrt{1-x^2}} \frac{e^y dy}{(1+e^y)\sqrt{1-x^2-y^2}}$$

and hence evaluate it with a rough figure.

S

(b) Prove that

$$\int_{0}^{1} \left\{ \int_{0}^{1} \frac{(x-y)}{(x+y)^3} \, dy \right\} dx =$$

211

but the value changes its sign as the order of integration interchange.

ŝ

UNIT-V

Show that $f_n(x) = e^{-nx}$ is pointwise but not uniformly convergent in [0,]. Also show that the convergence is uniform in [k,], where k is any positive number. **9**.

ŝ

that to converge uniformly to f on [a, b] if and only if to each $\varepsilon > 0$ and Show that a sequence of function $\{f_n(x)\}$ defined on [a, b] is said such Ę integer an $|f_{n+p}(x) - f_n(x)| < \varepsilon \,\forall n \ge m, \ p \ge 1.$ exists there $\forall x \in [a, b]$, (q)

ŝ

8 0 8

10. (a) Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3 + n^4 x^2}$$

0

ŝ is uniformly convergent for all values of x and that the derivative of the sum with respect to x is given by term-by-term differentiation

3 Examine the term-by-term integration of the series whose sum to first n terms is $\sum_{(n+x^2)^2}$. (q)

/543

24G-220

MATH/VI/CC/10

Student's Copy

2024

(CBCS)

(6th Semester)

MATHEMATICS

TENTH PAPER

(Advanced Calculus)

Full Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

(SECTION : A OBJECTIVE)

(Marks : 10)

 $1 \times 10 = 10$ Put a Tick \square mark against the correct answer in the boxes provided :

1. If f(x) = x over [0, 1] and $P = \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, \frac{n}{n} = 1\right\}$ be the partition,

then the value of the lower Riemann integral $\int_0^1 f \, dx$ is

(a)
$$\frac{1}{3}$$
 \Box (b) $\frac{1}{2}$ \Box

(d) None of the above (c) $-\frac{1}{2}$

- **2.** For any two partitions P_1, P_2 of the domain of a bounded function f (b) $L(P_1, f) < U(P_2, f)$ (a) $L(P_1, f) \leq U(P_2, f)$

 - (d) $U(P_2, f) \le L(P_1, f)$ (c) $L(P_2, f) \leq U(P_1, f)$

/543

-

Contd.

ю.	If f and g be two posit	ive functions such the	3. If f and g be two positive functions such that $f(x) \le g(x) \forall x \in [\alpha, b]$, then
	(a) $\int_a^b g dx$ converges if $\int_a^b f dx$ converges	if $\int_a^b f dx$ converges	
	(b) $\int_a^b f dx$ converges if $\int_a^b g dx$ converges	i if $\int_a^b g dx$ converges	
	(c) $\int_a^b f dx$ diverges if $\int_a^b g dx$ diverges	f $\int_a^b g \ dx$ diverges	
	(d) None of the above	ve	
4	4. The value of the improper integral	proper integral	
	$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx$	- dx	
	(a) is $\frac{\pi}{2} \log\left\{\frac{a}{b}\right\}$	(q)	(b) is $\log\left\{\frac{a}{b}\right\}$ \Box
	(c) is $\log\left\{\frac{b}{a}\right\}$		(d) does not exist
ທ່	The value of the im	proper integral $\int_0^\infty e^{-x}$	5. The value of the improper integral $\int_0^\infty e^{-x^2} \cos \alpha x dx$ is equal to
	(a) $\frac{\sqrt{\pi}}{2}$	(q)	(b) $\frac{\sqrt{\pi}}{2}e^{-\frac{\alpha}{4}}$
	(c) $\frac{\pi}{\sqrt{2}}e^{-\frac{\alpha^2}{4}}$		(d) $\frac{\sqrt{\pi}}{2}e^{-\frac{\alpha^2}{4}}$
é.	The value of the imp	roper integral $\int_0^{\pi} \frac{dx}{a+b cc}$	6. The value of the improper integral $\int_0^{\pi} \frac{dx}{a + b \cos x}$, if a is positive and $ b < a$
	is $(a) \frac{2\pi}{(a^2 - b^2)^{1/2}}$	(q)	$\frac{2\pi}{(a^2-b^2)^{3/2}}$
	(c) $\frac{\pi}{(a^2 - b^2)^{1/2}}$	(p)	$\frac{\pi}{(a^2-b^2)^{3/2}}$
/543	_	0	[Contd.

Contd.

c

7. The value of the double integral $\int_{1}^{1} \int_{1}^{1} \frac{x-y}{x+y} dx dy$ is

517 q 0 (a)

101 Ð 210 <u>(</u>)

curve the 13 C where qx $c_{C}x+y$ integral $x = at^2$, y = 2at, $0 \le t \le 2$, is the q value 8. The

log 4 (q) -19 (a)

log 2 (q) -10 (c)

9. By M_n -test, the sequence $\{f_n\}$ converges uniformly to f on [a, b] if and only if

 $M_n \to \infty$ as $n \to 0$ (a) $M_n \to 0 \text{ as } n \to 0$

(c) $M_n \to 0 \text{ as } n \to \infty$ (q

None of the above (q)

15 u + x10. The sequence of function given by $f_n(x) = \frac{n}{\dots}$ uniformly convergent in [0, k], whatever k may be (a)

only pointwise convergent in [0, k], whatever k may be (q)

non-convergent at all in [0, k], whatever k may be <u>ی</u>

uniformly convergent in $[0, \infty)$ (q)

/543

(<i>Marks</i> : 15) (<i>Marks</i> : 15) (<i>Marks</i> : 15) Answer the following questions : UNT-I 1. If a function f is monotonic on $[a, b]$, then prove that it is R-integrable over [a, b]. OR 2. Let f be a real-valued function defined by $f(x) = \begin{cases} 1, \text{ when } x \text{ is rational number} \\ -1, \text{ when } x \text{ is irrational number} \end{cases}$ Prove that $ f $ is R-integrable on any interval $[a, b]$ while f is not R-integrable. UNT-II 3. Using Dirichlet's test, prove that $\int_{0}^{\infty} \frac{\cos x}{\sqrt{a}} dx$ is convergent at e . OR 4. Prove that the improper integral $\int_{a}^{b} f dx$ converges at a if and only if to every $\varepsilon > 0$, there corresponds $\delta > 0$ such that $\int_{a+\lambda_{1}}^{f} f dx \Big _{c} < c_{0} < \lambda_{1}, \lambda_{2} < \delta$.	UNITIII Establish the uniform convergence of the improper integral $\int_0^\infty \frac{y}{x^2 + y^2} dx, (0 < c \le y \le d)$
---	--

[Contd.

4

/543

S

Prove that a necessary and sufficient condition for the integrability of bounded function f is that to every $\varepsilon > 0$, there exists a corresponding $\delta > 0$ such that $U(P, f) - L(P, f) < \varepsilon$ for every partition P of [a, b] with norm $\mu(P) < \delta$. 1. (a)

 $10 \times 5 = 50$

Answer the following questions

- UNIT-I

/543

g

$$\oint_{b, \text{ diven that }} \int_0^\infty \frac{\cos mx}{a^2 + x^2} dx = \frac{\pi}{2a} e^{-ma}. \text{ Prove that } \int_0^\infty \frac{\sin mx}{x(1 + x^2)} dx = \frac{\pi}{2} (1 - e^{-max}) e^{-max} dx = \frac{$$

÷

UNIT-IV

7. Show that

$$\int_0^{\pi} \int_0^{\sin x} y dy dx = \frac{\pi}{2}$$

g

- circle the 8. Show that $\int_C [(x-y)^3 dx + (x-y)^3 dy] = 3\pi a^4$, taken along
 - $x^2 + y^2 = a^2$ in the counter-clockwise sense.

UNIT-V

9. Prove that the sequence $f_n(\mathbf{x}) = x^n$ is pointwise convergent on [0, 1] and evaluate the pointwise limit.

ğ

10. Show that $f_n(x) = \frac{1}{x+n}$ is uniformly convergent in any interval [0, b], where

(SECTION : C-DESCRIPTIVE)

(Marks : 50)

b > 0.

21/2+21/2=5 ŝ 5 **2.** (a) If $\int_a^b f dx$ and $\int_a^b g dx$ both exist and g(x) keeps the same sign over $[\alpha, b]$, then prove that there exists a number μ lying between the (b) Compute the value of $\int_{-1}^{1} f dx$, where f(x) = |x| by dividing the interval converges absolutely for p > 1 but only conditionally for 0 .(b) Making use of the Riemann sum S(P, f), show that 3. (a) Examine the convergence of the following functions : $\int_{a}^{b} f(x)g(x) \, dx = \mu \int_{a}^{b} g \, dx$ $\int_1^2 f(x)dx = \frac{11}{2}$ $\int_{1}^{\infty} \frac{\sin x}{x^{p}} dx$ UNIT-II [-1, 1] into 2n equal sub-intervals. is convergent if and only if n > 0. g g bounds of f such that 4. (a) Prove that the integral (ii) $\int_0^\infty \frac{x \tan^{-1} x}{(1+x^4)^{1/3}} dx$ where f(x) = 3x + 1. $\int_1^\infty x^{n-1} e^{-x} dx$ (i) $\int_{2}^{\infty} \frac{2x^2}{x^4 - 1} dx$ (b) Show that

ŝ

ŝ

S

/543

ø

Contd.

(b) If $\phi(x)$ is bounded and monotonic in $[a, \infty)$ and $\int_a^\infty f(x) dx$ is convergent at

 ∞ , then prove that $\int_a^\infty f(x)\phi(x)dx$ is convergent at ∞ .

UNIT-III

- S prove that uniformly convergent improper integral of a continuous function is itself continuous. 5. (a)
- (b) Show that

$$\int_0^{\pi/2} \log(a^2 \cos^2 \theta + b^2 \sin^2 \theta) d\theta = \pi \log\left(\frac{a+b}{2}\right)$$

g

6. (a) Let $\phi(y) = \int_a^b f(x, y) dx$, is continuous and f_y also exists and continuous in [a, b; c, d]. Prove that ϕ is derivable and $\phi'(y) = \int_a^b f_y(x, y) dx \quad \forall y \in [c, d]$.

ŝ

(b) Prove that the function

$$\phi(y) = \int_{a}^{b} f(x, y) \, dx$$

is continuous in [c, d].

7. (a) Evaluate the integral

$$\int_{0}^{8} \int_{\overline{y}}^{2} \sqrt{x^{4} + 1} dx dy$$

by reversing the order of integration with a rough figure.

(b) Show that if 0 < h < 1

$$\operatorname{But} \int_{0}^{1} \left\{ \int_{0}^{1} f(x, y) dx \right\} dy = \int_{h}^{1} \left\{ \int_{h}^{1} f(x, y) dy \right\} dx = 0$$

But
$$\int_{0}^{1} \left\{ \int_{0}^{1} f(x, y) dx \right\} dy \neq \int_{0}^{1} \left\{ \int_{0}^{1} f(x, y) dy \right\} dx, \text{ where } f(x, y) = \frac{y^{2} - x^{2}}{(y^{2} + x^{2})^{2}}.$$

/543

2

Contd.

ŝ

ŝ

S

g

Change the order of integration in the integral (a) ø

$$\int_{0}^{1} dx \int_{0}^{\sqrt{1-x^2}} \frac{e^y dy}{(1+e^y)\sqrt{1-x^2-y^2}}$$

and hence evaluate it with a rough figure.

ŝ

(b) Prove that

$$\int_{0}^{1} \left\{ \int_{0}^{1} \frac{(x-y)}{(x+y)^3} \, dy \right\} dx = \frac{1}{2}$$

but the value changes its sign as the order of integration interchange.

UNIT-V

- Show that $f_n(x) = e^{-nx}$ is pointwise but not uniformly convergent in $[0, \infty]$. Also show that the convergence is uniform in $[k, \infty]$, where k is any positive number. 9. (a)
- that Show that a sequence of function $\{f_n(x)\}$ defined on [a, b] is said to converge uniformly to f on [a, b] if and only if to each $\varepsilon > 0$ and such E integer an $|f_{n+p}(x)-f_n(x)|<\varepsilon\,\forall n\geq m,\ p\geq 1.$ exists there $\forall x \in [a, b],$ (q)

0R

10. (a) Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3 + n^4 x^2}$$

is uniformly convergent for all values of x and that the derivative of the sum with respect to x is given by term-by-term differentiation.

S

Examine the term-by-term integration of the series whose sum to first × n te (q)

rms is
$$\sum \frac{x}{(n+x^2)^2}$$
.

/543

24G-220

œ

S

S

S