2 0 2 4

(CBCS)

(6th Semester)

MATHEMATICS

TENTH PAPER

(Advanced Calculus)

Full Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

(SECTION : A—OBJECTIVE)

(*Marks* : 10)

Put a Tick ☑ mark against the correct answer in the boxes provided : 1×10=10

1. If $f(x)=x$ over $[0,1]$ and $P=\left\{0, \frac{1}{n}, \frac{2}{n}, \cdots\cdots \frac{n-1}{n}, \frac{n}{n}=1\right\}$ be the partition, then the value of the lower Riemann integral $\underline{\int_0^1} f\,dx$ is

(a) $\frac{1}{3}$ ☐ (b) $\frac{1}{2}$ ☐

(c) $-\frac{1}{2}$ ☐ (d) None of the above ☐

2. For any two partitions P_1, P_2 of the domain of a bounded function f

(a) $L(P_1, f) \leq U(P_2, f)$ ☐ (b) $L(P_1, f) < U(P_2, f)$ ☐

(c) $L(P_2, f) \leq U(P_1, f)$ ☐ (d) $U(P_2, f) \leq L(P_1, f)$ ☐

3. If f and g be two positive functions such that $f(x) \le g(x) \;\forall x \in [a, b]$, then

(a) $\int_a^b g\,dx$ converges if $\int_a^b f\,dx$ converges ☐

(b) $\int_a^b f\,dx$ converges if $\int_a^b g\,dx$ converges ☐

(c) $\int_a^b f\,dx$ diverges if $\int_a^b g\,dx$ diverges ☐

(d) None of the above ☐

4. The value of the improper integral

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx$$

(a) is $\frac{\pi}{2}\log\left\{\frac{a}{b}\right\}$ ☐

(b) is $\log\left\{\frac{a}{b}\right\}$ ☐

(c) is $\log\left\{\frac{b}{a}\right\}$ ☐

(d) does not exist ☐

5. The value of the improper integral $\int_0^\infty e^{-x^2} \cos\alpha x\,dx$ is equal to

(a) $\frac{\sqrt{\pi}}{2}$ ☐

(b) $\frac{\sqrt{\pi}}{2} e^{-\frac{\alpha}{4}}$ ☐

(c) $\frac{\pi}{\sqrt{2}} e^{-\frac{\alpha^2}{4}}$ ☐

(d) $\frac{\sqrt{\pi}}{2} e^{-\frac{\alpha^2}{4}}$ ☐

6. The value of the improper integral $\int_0^\pi \frac{dx}{a + b\cos x}$, if a is positive and $|b| < a$ is

(a) $\frac{2\pi}{(a^2 - b^2)^{1/2}}$ ☐

(b) $\frac{2\pi}{(a^2 - b^2)^{3/2}}$ ☐

(c) $\frac{\pi}{(a^2 - b^2)^{1/2}}$ ☐

(d) $\frac{\pi}{(a^2 - b^2)^{3/2}}$ ☐

7. The value of the double integral $\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}\frac{x-y}{x+y}\,dxdy$ is

(a) 0 ☐

(b) $-\frac{1}{2}$ ☐

(c) $\frac{1}{2}$ ☐

(d) $\frac{\pi}{2}$ ☐

8. The value of the integral $\int_C \frac{dx}{x+y}$, where C is the curve $x = at^2$, $y = 2at$, $0 \le t \le 2$, is

(a) $\frac{1}{6}$ ☐

(b) $\log 4$ ☐

(c) $-\frac{1}{6}$ ☐

(d) $\log 2$ ☐

9. By M_n-test, the sequence $\{f_n\}$ converges uniformly to f on $[a, b]$ if and only if

(a) $M_n \to 0$ as $n \to 0$ ☐

(b) $M_n \to \infty$ as $n \to 0$ ☐

(c) $M_n \to 0$ as $n \to \infty$ ☐

(d) None of the above ☐

10. The sequence of function given by $f_n(x) = \frac{n}{x+n}$ is

(a) uniformly convergent in $[0, k]$, whatever k may be ☐

(b) only pointwise convergent in $[0, k]$, whatever k may be ☐

(c) non-convergent at all in $[0, k]$, whatever k may be ☐

(d) uniformly convergent in $[0, \infty)$ ☐

(SECTION : B—SHORT ANSWERS)

(*Marks* : 15)

Answer the following questions : 3×5=15

UNIT—I

1. If a function f is monotonic on $[a, b]$, then prove that it is R-integrable over $[a, b]$.

OR

2. Let f be a real-valued function defined by

$$f(x) = \begin{cases} 1, & \text{when } x \text{ is rational number} \\ -1, & \text{when } x \text{ is irrational number} \end{cases}$$

Prove that $|f|$ is R-integrable on any interval $[a, b]$ while f is not R-integrable.

UNIT—II

3. Using Dirichlet's test, prove that

$$\int_0^\infty \frac{\cos x}{x} dx$$

is convergent at ∞.

OR

4. Prove that the improper integral $\int_a^b f\, dx$ converges at a if and only if to every $\varepsilon > 0$, there corresponds $\delta > 0$ such that $\left| \int_{a+\lambda_1}^{a+\lambda_2} f\, dx \right| < \varepsilon, 0 < \lambda_1, \lambda_2 < \delta$.

UNIT—III

5. Establish the uniform convergence of the improper integral

$$\int_0^\infty \frac{y}{x^2 + y^2} dx, (0 < c \le y \le d)$$

OR

6. Given that $\int_0^\infty \frac{\cos mx}{a^2 + x^2} dx = \frac{\pi}{2a} e^{-ma}$. Prove that $\int_0^\infty \frac{\sin mx}{x(1 + x^2)} dx = \frac{\pi}{2}(1 - e^{-m})$.

UNIT—IV

7. Show that

$$\int_0^\pi \int_0^{\sin x} y dy dx = \frac{\pi}{2}$$

OR

8. Show that $\int_C [(x-y)^3 dx + (x-y)^3 dy] = 3\pi a^4$, taken along the circle $x^2 + y^2 = a^2$ in the counter-clockwise sense.

UNIT—V

9. Prove that the sequence $f_n(x) = x^n$ is pointwise convergent on $[0, 1]$ and evaluate the pointwise limit.

OR

10. Show that $f_n(x) = \frac{1}{x + n}$ is uniformly convergent in any interval $[0, b]$, where $b > 0$.

(SECTION : C—DESCRIPTIVE)

(*Marks* : 50)

Answer the following questions : 10×5=50

UNIT—I

1. *(a)* Prove that a necessary and sufficient condition for the integrability of bounded function f is that to every $\varepsilon > 0$, there exists a corresponding $\delta > 0$ such that $U(P, f) - L(P, f) < \varepsilon$ for every partition P of $[a, b]$ with norm $\mu(P) < \delta$. 5

(b) Making use of the Riemann sum $S(P, f)$, show that

$$\int_1^2 f(x)dx = \frac{11}{2}$$

where $f(x) = 3x + 1$. 5

OR

2. (a) If $\int_a^b f\,dx$ and $\int_a^b g\,dx$ both exist and $g(x)$ keeps the same sign over $[a, b]$, then prove that there exists a number μ lying between the bounds of f such that

$$\int_a^b f(x)g(x)\,dx = \mu \int_a^b g\,dx$$ 5

(b) Compute the value of $\int_{-1}^1 f\,dx$, where $f(x) = |x|$ by dividing the interval $[-1, 1]$ into $2n$ equal sub-intervals. 5

UNIT—II

3. (a) Examine the convergence of the following functions : 2½+2½=5

(i) $\int_2^\infty \frac{2x^2}{x^4 - 1}dx$

(ii) $\int_0^\infty \frac{x\tan^{-1} x}{(1 + x^4)^{1/3}}dx$

(b) Show that

$$\int_1^\infty \frac{\sin x}{x^p}dx$$

converges absolutely for $p > 1$ but only conditionally for $0 < p < 1$. 5

OR

4. (a) Prove that the integral

$$\int_1^\infty x^{n-1}e^{-x}dx$$

is convergent if and only if $n > 0$. 5

(b) If $\phi(x)$ is bounded and monotonic in $[a, \infty)$ and $\int_a^\infty f(x)dx$ is convergent at ∞, then prove that $\int_a^\infty f(x)\phi(x)dx$ is convergent at ∞. 5

UNIT—III

5. (a) Prove that uniformly convergent improper integral of a continuous function is itself continuous. 5

(b) Show that

$$\int_0^{\pi/2} \log(a^2 \cos^2\theta + b^2 \sin^2\theta)d\theta = \pi \log\left(\frac{a+b}{2}\right)$$ 5

OR

6. (a) Let $\phi(y) = \int_a^b f(x, y)dx$, is continuous and f_y also exists and continuous in $[a, b; c, d]$. Prove that ϕ is derivable and $\phi'(y) = \int_a^b f_y(x, y)dx \ \forall y \in [c, d]$. 5

(b) Prove that the function

$$\phi(y) = \int_a^b f(x, y)\, dx$$

is continuous in $[c, d]$. 5

UNIT—IV

7. (a) Evaluate the integral

$$\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4 + 1}dxdy$$

by reversing the order of integration with a rough figure. 5

(b) Show that if $0 < h < 1$

$$\int_h^1 \left\{\int_h^1 f(x, y)dx\right\} dy = \int_h^1 \left\{\int_h^1 f(x, y)dy\right\} dx = 0$$

But $\int_0^1 \left\{\int_0^1 f(x, y)dx\right\} dy \neq \int_0^1 \left\{\int_0^1 f(x, y)dy\right\} dx$, where $f(x, y) = \frac{y^2 - x^2}{(y^2 + x^2)^2}$. 5

OR

8. *(a)* Change the order of integration in the integral

$$\int_0^1 dx \int_0^{\sqrt{1-x^2}} \frac{e^y dy}{(1+e^y)\sqrt{1-x^2-y^2}}$$

and hence evaluate it with a rough figure. 5

(b) Prove that

$$\int_0^1 \left\{ \int_0^1 \frac{(x-y)}{(x+y)^3} dy \right\} dx = \frac{1}{2}$$

but the value changes its sign as the order of integration interchange. 5

UNIT—V

9. *(a)* Show that $f_n(x) = e^{-nx}$ is pointwise but not uniformly convergent in $[0, \infty[$. Also show that the convergence is uniform in $[k, \infty[$, where k is any positive number. 5

(b) Show that a sequence of function $\{f_n(x)\}$ defined on $[a, b]$ is said to converge uniformly to f on $[a, b]$ if and only if to each $\varepsilon > 0$ and $\forall x \in [a, b]$, there exists an integer m such that $|f_{n+p}(x) - f_n(x)| < \varepsilon \; \forall n \geq m, p \geq 1$. 5

OR

10. *(a)* Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3 + n^4 x^2}$$

is uniformly convergent for all values of x and that the derivative of the sum with respect to x is given by term-by-term differentiation. 5

(b) Examine the term-by-term integration of the series whose sum to first n terms is $\sum \frac{x}{(n+x^2)^2}$. 5

* * *

MATH/VI/CC/10 **Student's Copy**

2 0 2 4

(CBCS)

(6th Semester)

MATHEMATICS

TENTH PAPER

(Advanced Calculus)

Full Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

(SECTION : A—OBJECTIVE)

(*Marks* : 10)

Put a Tick ☑ mark against the correct answer in the boxes provided : 1×10=10

1. If $f(x) = x$ over $[0, 1]$ and $P = \left\{0, \frac{1}{n}, \frac{2}{n}, \cdots\cdots \frac{n-1}{n}, \frac{n}{n} = 1\right\}$ be the partition, then the value of the lower Riemann integral $\underline{\int_0^1} f \, dx$ is

(a) $\frac{1}{3}$ ☐ (b) $\frac{1}{2}$ ☐

(c) $-\frac{1}{2}$ ☐ (d) None of the above ☐

2. For any two partitions P_1, P_2 of the domain of a bounded function f

(a) $L(P_1, f) \leq U(P_2, f)$ ☐ (b) $L(P_1, f) < U(P_2, f)$ ☐

(c) $L(P_2, f) \leq U(P_1, f)$ ☐ (d) $U(P_2, f) \leq L(P_1, f)$ ☐

3. If f and g be two positive functions such that $f(x) \leq g(x) \ \forall x \in [a, b]$, then

(a) $\int_a^b g \, dx$ converges if $\int_a^b f \, dx$ converges ☐

(b) $\int_a^b f \, dx$ converges if $\int_a^b g \, dx$ converges ☐

(c) $\int_a^b f \, dx$ diverges if $\int_a^b g \, dx$ diverges ☐

(d) None of the above ☐

4. The value of the improper integral

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx$$

(a) is $\frac{\pi}{2} \log\left\{\frac{a}{b}\right\}$ ☐

(b) is $\log\left\{\frac{a}{b}\right\}$ ☐

(c) is $\log\left\{\frac{b}{a}\right\}$ ☐

(d) does not exist ☐

5. The value of the improper integral $\int_0^\infty e^{-x^2} \cos\alpha x \, dx$ is equal to

(a) $\frac{\sqrt{\pi}}{2}$ ☐

(b) $\frac{\sqrt{\pi}}{2} e^{-\frac{\alpha}{4}}$ ☐

(c) $\frac{\pi}{\sqrt{2}} e^{-\frac{\alpha^2}{4}}$ ☐

(d) $\frac{\sqrt{\pi}}{2} e^{-\frac{\alpha^2}{4}}$ ☐

6. The value of the improper integral $\int_0^\pi \frac{dx}{a + b\cos x}$, if a is positive and $|b| < a$ is

(a) $\frac{2\pi}{(a^2 - b^2)^{1/2}}$ ☐

(b) $\frac{2\pi}{(a^2 - b^2)^{3/2}}$ ☐

(c) $\frac{\pi}{(a^2 - b^2)^{1/2}}$ ☐

(d) $\frac{\pi}{(a^2 - b^2)^{3/2}}$ ☐

7. The value of the double integral $\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} \frac{x-y}{x+y}\,dxdy$ is

(a) 0 ☐ (b) $-\frac{1}{2}$ ☐

(c) $\frac{1}{2}$ ☐ (d) $\frac{\pi}{2}$ ☐

8. The value of the integral $\int_C \frac{dx}{x+y}$, where C is the curve $x = at^2, y = 2at, 0 \le t \le 2$, is

(a) $\frac{1}{6}$ ☐ (b) $\log 4$ ☐

(c) $-\frac{1}{6}$ ☐ (d) $\log 2$ ☐

9. By M_n-test, the sequence $\{f_n\}$ converges uniformly to f on $[a, b]$ if and only if

(a) $M_n \to 0$ as $n \to 0$ ☐

(b) $M_n \to \infty$ as $n \to 0$ ☐

(c) $M_n \to 0$ as $n \to \infty$ ☐

(d) None of the above ☐

10. The sequence of function given by $f_n(x) = \frac{n}{x+n}$ is

(a) uniformly convergent in $[0, k]$, whatever k may be ☐

(b) only pointwise convergent in $[0, k]$, whatever k may be ☐

(c) non-convergent at all in $[0, k]$, whatever k may be ☐

(d) uniformly convergent in $[0, \infty)$ ☐

(SECTION : B—SHORT ANSWERS)

(*Marks* : 15)

Answer the following questions : 3×5=15

UNIT—I

1. If a function f is monotonic on $[a, b]$, then prove that it is R-integrable over $[a, b]$.

OR

2. Let f be a real-valued function defined by

$$f(x) = \begin{cases} 1, & \text{when } x \text{ is rational number} \\ -1, & \text{when } x \text{ is irrational number} \end{cases}$$

Prove that $|f|$ is R-integrable on any interval $[a, b]$ while f is not R-integrable.

UNIT—II

3. Using Dirichlet's test, prove that

$$\int_0^\infty \frac{\cos x}{x}\, dx$$

is convergent at ∞.

OR

4. Prove that the improper integral $\int_a^b f\, dx$ converges at a if and only if to every $\varepsilon > 0$, there corresponds $\delta > 0$ such that $\left| \int_{a+\lambda_1}^{a+\lambda_2} f\, dx \right| < \varepsilon,\ 0 < \lambda_1, \lambda_2 < \delta$.

UNIT—III

5. Establish the uniform convergence of the improper integral

$$\int_0^\infty \frac{y}{x^2 + y^2}\, dx, \ (0 < c \le y \le d)$$

OR

6. Given that $\int_0^\infty \frac{\cos mx}{a^2+x^2}dx = \frac{\pi}{2a}e^{-ma}$. Prove that $\int_0^\infty \frac{\sin mx}{x(1+x^2)}dx = \frac{\pi}{2}(1-e^{-m})$.

UNIT—IV

7. Show that

$$\int_0^\pi \int_0^{\sin x} y\,dy\,dx = \frac{\pi}{2}$$

OR

8. Show that $\int_C [(x-y)^3 dx + (x-y)^3 dy] = 3\pi a^4$, taken along the circle $x^2+y^2=a^2$ in the counter-clockwise sense.

UNIT—V

9. Prove that the sequence $f_n(x) = x^n$ is pointwise convergent on $[0, 1]$ and evaluate the pointwise limit.

OR

10. Show that $f_n(x) = \frac{1}{x+n}$ is uniformly convergent in any interval $[0, b]$, where $b > 0$.

(**SECTION : C—DESCRIPTIVE**)

(*Marks* : 50)

Answer the following questions : $10\times5=50$

UNIT—I

1. (a) Prove that a necessary and sufficient condition for the integrability of bounded function f is that to every $\varepsilon > 0$, there exists a corresponding $\delta > 0$ such that $U(P, f) - L(P, f) < \varepsilon$ for every partition P of $[a, b]$ with norm $\mu(P) < \delta$. 5

(b) Making use of the Riemann sum $S(P, f)$, show that

$$\int_1^2 f(x)dx = \frac{11}{2}$$

where $f(x) = 3x + 1$. 5

OR

2. (a) If $\int_a^b f\, dx$ and $\int_a^b g\, dx$ both exist and $g(x)$ keeps the same sign over $[a, b]$, then prove that there exists a number μ lying between the bounds of f such that

$$\int_a^b f(x)g(x)\, dx = \mu \int_a^b g\, dx$$ 5

(b) Compute the value of $\int_{-1}^{1} f\, dx$, where $f(x) = |x|$ by dividing the interval $[-1, 1]$ into $2n$ equal sub-intervals. 5

UNIT—II

3. (a) Examine the convergence of the following functions : 2½+2½=5

(i) $\int_2^\infty \frac{2x^2}{x^4 - 1} dx$

(ii) $\int_0^\infty \frac{x \tan^{-1} x}{(1 + x^4)^{1/3}} dx$

(b) Show that

$$\int_1^\infty \frac{\sin x}{x^p} dx$$

converges absolutely for $p > 1$ but only conditionally for $0 < p < 1$. 5

OR

4. (a) Prove that the integral

$$\int_1^\infty x^{n-1} e^{-x} dx$$

is convergent if and only if $n > 0$. 5

(b) If $\phi(x)$ is bounded and monotonic in $[a, \infty)$ and $\int_a^\infty f(x)dx$ is convergent at ∞, then prove that $\int_a^\infty f(x)\phi(x)dx$ is convergent at ∞. 5

UNIT—III

5. (a) Prove that uniformly convergent improper integral of a continuous function is itself continuous. 5

(b) Show that

$$\int_0^{\pi/2} \log(a^2\cos^2\theta + b^2\sin^2\theta)d\theta = \pi\log\left(\frac{a+b}{2}\right)$$ 5

OR

6. (a) Let $\phi(y) = \int_a^b f(x, y)dx$, is continuous and f_y also exists and continuous in $[a, b; c, d]$. Prove that ϕ is derivable and $\phi'(y) = \int_a^b f_y(x, y)dx \ \forall y \in [c, d]$. 5

(b) Prove that the function

$$\phi(y) = \int_a^b f(x, y)\,dx$$

is continuous in $[c, d]$. 5

UNIT—IV

7. (a) Evaluate the integral

$$\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4 + 1}\,dx\,dy$$

by reversing the order of integration with a rough figure. 5

(b) Show that if $0 < h < 1$

$$\int_h^1 \left\{\int_h^1 f(x, y)dx\right\}dy = \int_h^1 \left\{\int_h^1 f(x, y)dy\right\}dx = 0$$

But $\int_0^1 \left\{\int_0^1 f(x, y)dx\right\}dy \neq \int_0^1 \left\{\int_0^1 f(x, y)dy\right\}dx$, where $f(x, y) = \dfrac{y^2 - x^2}{(y^2 + x^2)^2}$. 5

OR

8. *(a)* Change the order of integration in the integral

$$\int_0^1 dx \int_0^{\sqrt{1-x^2}} \frac{e^y dy}{(1+e^y)\sqrt{1-x^2-y^2}}$$

and hence evaluate it with a rough figure. 5

(b) Prove that

$$\int_0^1 \left\{ \int_0^1 \frac{(x-y)}{(x+y)^3} dy \right\} dx = \frac{1}{2}$$

but the value changes its sign as the order of integration interchange. 5

UNIT—V

9. *(a)* Show that $f_n(x) = e^{-nx}$ is pointwise but not uniformly convergent in $[0, \infty[$. Also show that the convergence is uniform in $[k, \infty[$, where k is any positive number. 5

(b) Show that a sequence of function $\{f_n(x)\}$ defined on $[a, b]$ is said to converge uniformly to f on $[a, b]$ if and only if to each $\varepsilon > 0$ and $\forall x \in [a, b]$, there exists an integer m such that $|f_{n+p}(x) - f_n(x)| < \varepsilon \; \forall n \geq m, p \geq 1$. 5

OR

10. *(a)* Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3 + n^4 x^2}$$

is uniformly convergent for all values of x and that the derivative of the sum with respect to x is given by term-by-term differentiation. 5

(b) Examine the term-by-term integration of the series whose sum to first n terms is $\sum \frac{x}{(n+x^2)^2}$. 5

* * *