Student's Copy

2024

(CBCS)

(6th Semester)

MATHEMATICS

TWELFTH (B) PAPER

(Elementary Number Theory)

Full Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

(SECTION : A—OBJECTIVE)

(*Marks* : 10)

Tick (✓) the correct answer in the brackets provided : $1\times10=10$

1. The greatest common divisor of 12378 and 3054 is

(a) 6 ( )

(b) 12 ( )

(c) 16 ( )

(d) 3 ( )

2. The least common multiple of 1687 and 482 is

(a) 1687 ( )

(b) 2374 ( )

(c) 3374 ( )

(d) 6748 ( )

3. The solution of the linear congruence $16x \equiv 25 \pmod{19}$, where $t \in \mathbb{Z}$, is

(a) $x = 16 + 19t$ ( )

(b) $x = 17 + 19t$ ( )

(c) $x = 18 + 28t$ ( )

(d) $x = 20 + 28t$ ( )

4. The reduced residue system modulo 12 from the following is

(a) $\{1, 2, 3, 5, 7, 11\}$ ( )

(b) $\{1, 3, 5, 7, 11\}$ ( )

(c) $\{1, 5, 7, 11\}$ ( )

(d) $\{5, 7, 11\}$ ( )

5. The remainder when $3^{12} + 5^{12}$ is divisible by 13 is

(a) 0 ( )

(b) 1 ( )

(c) 2 ( )

(d) 3 ( )

6. For any integer $n > 2$, Euler's function $\phi(n)$ is

(a) zero ( )

(b) odd ( )

(c) even ( )

(d) None of the above ( )

7. If p is an odd prime and g.c.d. $(a, p) = 1$, then

(a) $\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod{p}$ ( )

(b) $\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod{a}$ ( )

(c) $\left(\frac{a}{p}\right) \equiv a^{\frac{p}{2}} \pmod{p}$ ( )

(d) $\left(\frac{a}{p}\right) \equiv a^{p-1} \pmod{p}$ ( )

8. If p is an odd prime and if g is a primitive root, then

(a) $\left(\frac{g}{p}\right) = 0$ ( )

(b) $\left(\frac{g}{p}\right) = 1$ ( )

(c) $\left(\frac{g}{p}\right) = -1$ ( )

(d) $\left(\frac{g}{p}\right) = p$ ( )

9. If f is multiplicative and $f(n) \neq z(n)$, then

(a) $f(1) = -1$ ( )

(b) $f(1) = 0$ ( )

(c) $f(1) = 3$ ( )

(d) $f(1) = 1$ ( )

10. The value of $\sigma(20)$ is

(a) 42 ( )

(b) 20 ( )

(c) 40 ( )

(d) 35 ( )

(SECTION : B—SHORT ANSWERS)

(*Marks* : 15)

Answer the following : $3 \times 5 = 15$

UNIT—I

1. Prove that if $a|b$ and $b|c$, then $a|c$.

OR

2. Find the g.c.d. of 275 and 200, and express it in the form $m \cdot 275 + n \cdot 200$.

UNIT—II

3. Show that $3 \times 5^{2n+1} + 2^{3n+1} \equiv 0 \pmod{17}$.

OR

4. If $a \equiv b \pmod{m}$ and d is a positive divisor of m, then show that $a \equiv b \pmod{d}$.

UNIT—III

5. Prove that $n^5 - n$ is divisible by 30.

OR

6. Show that if $n > 1$ and $p_1, p_2, \ldots, p_m$ are the distinct prime factor of n, then

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_m}\right)$$

UNIT—IV

7. Evaluate $\left(\frac{30}{43}\right)$.

OR

8. If the prime $p > 3$, $p \equiv 3 \pmod{4}$, and $q = 2p + 1$ is prime, then prove that $2^p - 1$ is composite.

Unit—V

9. Prove that for a Dirichlet product of arithmetic functions f and g, we have $f * g = g * f$.

OR

10. Evaluate $\phi(735)$.

(Section : C—Descriptive)

(*Marks* : 50)

Answer the following : $10 \times 5 = 50$

Unit—I

1. *(a)* State and prove division algorithm. 6

(b) If p is a prime and a, b are any integers, prove that $p | ab \Rightarrow p | a$ or $p | b$. 4

OR

2. *(a)* State and prove the fundamental theorem of arithmetic. 6

(b) For positive integers a and b, prove that g.c.d. $(a, b) \cdot$ l.c.m. $(a, b) = ab$. 4

Unit—II

3. *(a)* Prove that there are arbitrary large gaps in the series of primes, that is, given any positive integer k, there exists k consecutive composite integers. 5

(b) If a and b be two arbitrary integers, prove that $a \equiv b \pmod{m}$ if and only if a and b have the same remainder when divided by m. 5

OR

4. *(a)* Prove that the congruence $ax \equiv b \pmod{m}$ has a solution if and only if the greatest common divisor of a and m divides b. 6

(b) Prove that the number of positive primes is infinite. 4

UNIT—III

5. *(a)* State and prove Euler's theorem. 6

(b) If $n > 1$, then show that the sum of all positive integers which are less than n and prime to n is $\frac{1}{2}n\phi(n)$. 4

OR

6. *(a)* If p is prime, then prove that $(p-1)! \equiv -1 \pmod{p}$. 5

(b) Show that $16! + 86$ is divisible by 323 using Wilson's theorem. 5

UNIT—IV

7. *(a)* Let p and q be distinct odd primes. If $p \equiv q \equiv 3 \pmod 4$, then prove that $\left(\frac{q}{p}\right) = -\left(\frac{p}{q}\right)$. Otherwise, if $p \equiv 1 \pmod 4$ or $q \equiv 1 \pmod 4$, then show that $\left(\frac{q}{p}\right) = \left(\frac{p}{q}\right)$. 7

(b) Evaluate $\left(\frac{5}{13}\right)$. 3

OR

8. *(a)* Prove that if $2^n - 15 = x^2$, then $n = 4$ or $n = 6$. 7

(b) Evaluate $\left(\frac{5}{11}\right)$. 3

UNIT—V

9. *(a)* If f and g are both multiplicative functions, then prove that $f * g$ is also multiplicative. 6

(b) Prove that if $mn > 1$, then $\sigma(mn) > m\sigma(n)$. 4

OR

10. *(a)* State and prove Mobius inversion formula. 6

(b) Find the general solution of $10x - 8y = 42$. 4

★★★

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