Student's Copy

2024

(CBCS)

(6th Semester)

MATHEMATICS

NINTH PAPER

(Modern Algebra)

Full Marks : 75

Time : 3 hours

(SECTION : A—OBJECTIVE)

(*Marks* : 10)

Each question carries 1 mark

Put a Tick ☑ mark against the correct answer in the box provided :

1. Let G be a group with identity element e. Then G is called a simple group if

   (a) the only normal subgroup of G is $\{e\}$ ☐

   (b) the only normal subgroups of G are $\{e\}$ and G itself ☐

   (c) the only normal subgroup of G is G itself ☐

   (d) it has no normal subgroup ☐

2. A subgroup H of a group G is normal if it is of index

   (a) 0 ☐   *(b)* 1 ☐

   (c) 2 ☐   *(d)* infinity ☐

3. The set $(\{0, 1, 2, 3, 4, 5\}, +_6, \times_6)$ is a

(a) commutative ring with zero divisors ☐

(b) non-commutative ring with zero divisors ☐

(c) commutative ring without zero divisors ☐

(d) non-commutative ring without zero divisors ☐

4. The set of all 2×2 matrices of the form $\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$, where a, b are integers, is

(a) a left ideal but not a right ideal in the ring of all 2×2 matrices with elements as integers ☐

(b) a right ideal but not a left ideal in the ring of all 2×2 matrices with elements as integers ☐

(c) neither a left ideal nor a right ideal in the ring of all 2×2 matrices with elements as integers ☐

(d) both left ideal and right ideal in the ring of all 2×2 matrices with elements as integers ☐

5. The units in the ring of Gaussian integers are

(a) 0 and 1 ☐

(b) 1 and -1 ☐

(c) $1, -1, i$ and $-i$ ☐

(d) i and $-i$ ☐

6. The associate(s) of the zero element of a commutative ring R with unity is/are

(a) all the non-zero elements of R ☐

(b) all the elements of R ☐

(c) all the units of R ☐

(d) the zero element itself only ☐

7. If F is a field of complex numbers, then the vectors (a_1, a_2) and (b_1, b_2) in $V_2(F)$ are linearly dependent if and only if

(a) $a_1b_2 - a_2b_1 = 0$ ☐

(b) $a_1b_2 + a_2b_1 = 0$ ☐

(c) $a_1a_2 - b_1b_2 = 0$ ☐

(d) $a_1a_2 + b_1b_2 = 0$ ☐

8. A subset S of a vector space $V(F)$ is a basis of $V(F)$ if

(a) S consists of linearly dependent vectors and S generates V ☐

(b) S consists of linearly independent vectors and S generates V ☐

(c) S consists of linearly dependent vectors and V is generated by some superset of S ☐

(d) S consists of linearly independent vectors and V is generated by some subset of S ☐

9. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T(x, y, z) = (x + y, y - z)$. Then

(a) kernel of $T = L\{(-1, 0, 1)\}$ ☐

(b) kernel of $T = L\{(-1, 1, -1)\}$ ☐

(c) kernel of $T = L\{(-2, 1, 1)\}$ ☐

(d) kernel of $T = L\{(-1, 1, 1)\}$ ☐

10. Let V be an n-dimensional vector space over the field F and W be an m-dimensional vector space over F. Then the vector space $L(U, V)$ of linear transformations from V into W is also finite-dimensional and is of dimension

(a) mn ☐

(b) m ☐

(c) n ☐

(d) $m + n$ ☐

(SECTION : B—SHORT ANSWERS)

(*Marks* : 15)

Each question carries 3 marks

Answer the following questions :

UNIT—1

1. Show that every quotient group of a cyclic group is cyclic but the converse is not true.

OR

2. Prove that a subgroup H of a group G is a normal subgroup of G if and only if each left coset of H in G is a right coset of H in G.

UNIT—2

3. Show that every ideal of a ring R is a subring of R but the converse need not be true.

OR

4. Let R be a commutative ring with unity. Show that every prime ideal of R is a maximal ideal of R.

UNIT—3

5. Prove that the necessary and sufficient condition for a non-zero element a in a Euclidean ring to be a unit is that $d(a) = d(1)$.

OR

6. Show that a unit in a ring R is not a zero divisor.

UNIT—4

7. Show that the kernel of a homomorphism of vector spaces is a subspace.

OR

8. Show that if two vectors are linearly dependent, one of them is a scalar multiple of the other.

UNIT—5

9. Show that the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (y, x)$ is a linear transformation.

OR

10. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x, y) = (4x - 2y, 2x + y)$. Find the matrix of T with respect to the ordered basis $\{(1, 1), (-1, 0)\}$.

(SECTION : C—DESCRIPTIVE)

(*Marks* : 50)

The figures in the margin indicate full marks for the questions

Answer the following questions : 10×5=50

UNIT—1

1. (a) If H is a normal subgroup of a group G and K is a normal subgroup of G containing H, then prove that $\dfrac{G}{K} \cong \dfrac{G/H}{K/H}$. 6

(b) Let Z be the centre of a group G. If $\dfrac{G}{Z}$ is cyclic, then prove that G is abelian. 4

OR

2. (a) Prove that the set $I(G)$ of all inner automorphisms of a group G is a normal subgroup of the group of its automorphisms and is isomorphic to the quotient group G/Z of G, where Z is the centre of G. 6

(b) Let f be an isomorphic mapping of a group G into a group G' and let $a \in G$. Show that the order of a is equal to the order of $f(a)$. 4

UNIT—2

3. (a) Prove that the ring of integers is a principal ideal domain. 5

(b) Prove that every finite integral domain is a field. 5

OR

4. *(a)* Let R be a commutative ring and S is an ideal of R. Then show that the ring of residue classes R/S is an integral domain if and only if S is a prime ideal. 6

(b) Show that a commutative ring with unity is a field if it has no proper ideal. 4

UNIT—3

5. *(a)* Show that every non-zero element of a Euclidean ring R is either a unit in R or can be written as a product of a finite number of prime elements of R. 6

(b) Let R be an integral domain with unity element 1. Show that two non-zero elements $a, b \in R$ are associates if and only if $a \mid b$ and $b \mid a$. 4

OR

6. *(a)* Let R be a Euclidean ring and a and b be any two elements in T, not both of which are zero. Then prove that a and b have a greatest common divisor d which can be expressed in the form $d = \lambda a + \mu b$, for some $\lambda, \mu \in R$. 6

(b) Let f be a homomorphism of a ring R into a ring R'. Show that the kernel of f is an ideal of R. 4

UNIT—4

7. *(a)* Prove that if $V(F)$ is a finite-dimensional vector space, then any two bases of V have the same number of elements. 5

(b) Show that the union of two subspaces is a subspace if and only if one is contained in the other. 5

OR

8. (a) If U and W are two subspaces of a finite-dimensional vector space $V(F)$, then prove that $\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$. 6

(b) Show that the vectors $(2, -3, 1)$, $(3, -1, 5)$, $(1, -4, 3)$ form a basis for $\mathbb{R}^3$. 4

UNIT—5

9. (a) Let V be a finite-dimensional vector space over the field F and let $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be an ordered basis for V. Let W be vector space over the same field and let $\beta_1, \beta_2, \cdots, \beta_n$ be any n-vectors in W. Then prove that there exists a unique linear transformation T from V into W such that $T(\alpha_i) = \beta_i$, $i = 1, 2, \cdots, n$. 6

(b) If the matrix of a linear transformation T on $V_2(C)$ with respect to the ordered basis $B = \{(1, 0), (0, 1)\}$ is $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then find the matrix of T with respect to the basis $B = \{(1, 1), (-1, 1)\}$. 4

OR

10. (a) Let V and W be vector spaces over the same field F and let T be a linear transformation from V into W. If V is finite-dimensional, then prove that

$$\text{rank } T + \text{nullity } T = \dim V$$

6

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by

$$T(x, y, z) = (x + y, x + 2y - z, y - z)$$

Determine the rank and nullity of T. 4

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