Student's Copy

2024

(CBCS)

(5th Semester)

MATHEMATICS

SEVENTH PAPER

(Complex Analysis)

Full Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

(SECTION : A—OBJECTIVE)

(*Marks* : 10)

Tick ☑ the correct answer in the box provided : 1×10=10

1. If the principal argument, Arg $z = 0$, then z lies on the

   (a) positive imaginary axis ☐

   (b) positive real axis ☐

   (c) negative imaginary axis ☐

   (d) negative real axis ☐

2. If w_k be n-th root of a non-zero complex number z, then the roots w_k can be given as

(a) $w_k = \frac{1}{r^{1/n}}\left[\cos\left(\frac{\theta+2k\pi}{n}\right)+i\sin\left(\frac{\theta+2k\pi}{n}\right)\right]$ ☐

(b) $w_k = r^{1/n}[\cos(\theta+2k\pi)+i\sin(\theta+2k\pi)]$ ☐

(c) $w_k = r^{1/n}\left[\cos\left(\frac{\theta+2k\pi}{n}\right)+i\sin\left(\frac{\theta+2k\pi}{n}\right)\right]$ ☐

(d) $w_k = \left[\cos\left(\frac{\theta+2k\pi}{n}\right)+i\sin\left(\frac{\theta+2k\pi}{n}\right)\right]$ ☐

3. If a function is differentiable at every point of the domain, then we say that a function is

(a) continuous in that domain ☐

(b) analytic in that domain ☐

(c) harmonic in that domain ☐

(d) constant in that domain ☐

4. Polar form of Cauchy-Riemann equation is

(a) $\frac{\partial u}{\partial r} = -\frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$ ☐

(b) $\frac{\partial u}{\partial r} = -\frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial u}{\partial \theta} = r\frac{\partial v}{\partial r}$ ☐

(c) $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$ ☐

(d) $\frac{\partial u}{\partial r} = r\frac{\partial v}{\partial \theta}, \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$ ☐

5. A power series $\Sigma a_n z^n$ is said to be absolutely convergent if

(a) the series $\Sigma|a_n z^n|$ converges ☐

(b) the series $\Sigma a_n z^n$ converges ☐

(c) the sequence of partial sum converges ☐

(d) None of the above ☐

6. The radius of convergence of the power series $\Sigma \frac{n!}{n^n} z^n$ is

(a) e ☐ (b) ∞ ☐

(c) 1 ☐ (d) 0 ☐

7. If the function $f(z)$ is an entire function, then the integral of $f(z)$ over a curve C

(a) depends along the path ☐

(b) is always 0 ☐

(c) depends only on the end points of C ☐

(d) None of the above ☐

8. If $f(z)$ is analytic in a simply connected domain D, and z_0 is any point on D and C is any simple closed curve enclosing the point z_0, then

$$\oint_C \frac{f(z)dz}{z - z_0}$$

equals

(a) $2\pi i f(z_0)$ ☐ (b) $\pi i f(z_0)$ ☐

(c) $\{\pi i f(z_0)\} / 2$ ☐ (d) 0 ☐

9. If $z = a$ is the singular point of $f(z)$, and a is the limit point of zeroes of $f(z)$, then a is

(a) removable singularity ☐

(b) pole ☐

(c) non-isolated essential singularity ☐

(d) isolated essential singularity ☐

10. If $f(z)$ is such that $f(a) = 0$, then the point a is called a/an

(a) singular point of the function $f(z)$ ☐

(b) non-isolated point of the function $f(z)$ ☐

(c) zero of the function $f(z)$ ☐

(d) isolated point of the function $f(z)$ ☐

(SECTION : B—SHORT ANSWERS)

(*Marks* : 15)

Answer the following :

$3 \times 5 = 15$

UNIT—I

1. For any complex numbers z_1 and z_2, prove that $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$ if and only if $z_1\bar{z}_2$ is purely imaginary.

OR

2. Prove that for any complex number z, $|z^2| = z\bar{z}$ where $\bar{z}$ is the conjugate of z.

UNIT—II

3. Show that the function $f(z) = xy + iy$ is everywhere continuous but not analytic.

OR

4. Show that $f(z) = |z|^2$ is not differentiable for any values of z.

UNIT—III

5. Examine the convergence of the series Σz^n.

OR

6. Show that the series $\Sigma n! z^n$ converges only at $z = 0$.

UNIT—IV

7. Evaluate $\int_C (x^2 - iy^2) dz$, where C is a continuous curve from (0, 0) to (2, 4).

OR

8. Evaluate $\int_C z^2 dz$, where C is a circle $|z| = 2$ in the region from $0 \le \theta \le \frac{\pi}{3}$.

UNIT—V

9. State and prove Taylor's theorem.

OR

10. Examine the nature of the function

$$f(z) = \frac{1}{\sin z - \cos z}$$

at $z = \frac{\pi}{4}$.

(SECTION : C—DESCRIPTIVE)

(*Marks* : 50)

Answer the following : 10×5=50

UNIT—I

1. *(a)* If z_1, z_2 and z_3 are the vertices of an isosceles triangle, right angled at the vertex z_2, then prove that $z_1^2 + 2z_2^2 + z_3^2 = 2z_2(z_1 + z_3)$. 5

   (b) Find the four fourth roots of $(-2\sqrt{3} - 2i)$. 5

OR

2. *(a)* Prove the triangular inequality of a complex number. 4

   (b) Show that the equation of the form $az\bar{z} + \bar{\beta}z + \beta\bar{z} + r = 0$ where a and r are real constants and $\beta\bar{\beta} - ar \geq 0$ represents a straight line if $a = 0$ and a circle if $a \neq 0$. 6

UNIT—II

3. *(a)* Prove that every differentiable function of complex numbers is continuous but not conversely. 5

   (b) Show that the function

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{(x^2 + y^2)}, \; z \neq 0 \text{ and } f(0) = 0$$

   is continuous but not analytic at $z = 0$ although C-R equations are satisfied at the point. 5

OR

4. (a) If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and $u - v = e^x(\cos y - \sin y)$, find $if(z)$ in terms of z. 6

(b) Show that $w = f(z) = u^2$ is harmonic if and only if u is constant. 4

UNIT—III

5. (a) Examine the behaviour of the power series

$$\Sigma \frac{z^n}{n}$$

on the circle of convergence. 6

(b) For what values of z, does the series

$$\Sigma \frac{1}{(z^2 + 1)^n}$$

convergent? Find its sum. 4

OR

6. (a) Test the convergence of the series

$$\Sigma \frac{(-1)^n}{z + n}$$

and find the condition for the convergence of the series. 6

(b) Find the radii of convergence of the following series : 2+2=4

(i) $\sum \frac{1}{n^n} z^n$

(ii) $\sum 2^{\sqrt{n}} z^n$

UNIT—IV

7. (a) Find the value of the integral $\int_0^{1+i} (x - y + ix^2) dx$ along

(i) a straight line from $z = 0$ to $z = 1 + i$

(ii) the real axis from $z = 0$ to $z = 1$ and then along a line parallel to the imaginary axis from $z = 1$ to $z = 1 + i$.

(b) Evaluate

$$\oint_C \frac{dz}{(z-1)(z-2)(z+4)}$$

where $C : |z| = 3$. 5

OR

8. (a) Prove that if $f(z)$ be analytic in a simply connected domain D, let z_0 be any point in D and C be any simple closed curve in D enclosing the point z_0, then

$$\oint_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$$ 6

(b) Evaluate

$$\oint_{3i}^{2+4i} [(2y + x^2)dx + (3x - y)dy]$$

along the path $x = 2t$, $y = t^2 + 3$. 4

UNIT—V

9. (a) State and prove Liouville's theorem. 6

(b) If a function is given by $f(z) = 1/(z^3 + z^5)$, then what are the singularities of $f(z)$? State the order of the poles. 4

OR

10. (a) Obtain the Taylor and Laurent series for the function

$$f(z) = \frac{z^2 - 1}{(z+3)(z+2)}$$

in the region (i) $|z| < 2$ (ii) $2 < |z| < 3$ (iii) $|z| > 3$. 6

(b) Examine the nature of singularities of the following functions : 2+2=4

(i) $\cot z$ at $z = \infty$

(ii) $\sec \frac{1}{z}$ at $z = 0$

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