Student's Copy

2 0 2 4

(CBCS)

(4th Semester)

MATHEMATICS

FOURTH PAPER

(Vector Calculus and Solid Geometry)

Full Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

(SECTION : A—OBJECTIVE)

(*Marks* : 10)

Tick (✓) the correct answer in the brackets provided : 1×10=10

1. The value of $[i \; k \; j]$ is

   (a) 1 ( )

   (b) 2 ( )

   (c) 0 ( )

   (d) −1 ( )

2. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are any three vectors, then

$$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b})$$

is

(a) 0 ( )

(b) 1 ( )

(c) -1 ( )

(d) $\vec{b}$ ( )

3. If $\vec{V} = (2x + y)\hat{i} - (3y + 2z)\hat{j} + (x + az)\hat{k}$ is solenoidal, then the value of the constant a is

(a) $2x$ ( )

(b) $2z$ ( )

(c) 1 ( )

(d) 0 ( )

4. If $\phi(x, y, z) = x^2 + y^2 + z^2$, then the value of grad ϕ at the point (1, 2, 3) is

(a) $2\hat{i} + 4\hat{j} + 5\hat{k}$ ( )

(b) $\hat{i} + 4\hat{j} + 6\hat{k}$ ( )

(c) $2\hat{i} + 3\hat{j} + 6\hat{k}$ ( )

(d) $2\hat{i} + 4\hat{j} + 6\hat{k}$ ( )

5. By changing the origin to (2, 3) without changing the direction of axes, the equation $5x + 3y = 3$ changes to

(a) $5x + 3y = 1$ ( )

(b) $5x + 3y + 16 = 0$ ( )

(c) $5x + 3y + 6 = 0$ ( )

(d) None of the above ( )

6. The equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a circle, if

(a) $ab - h^2 = 0$ ( )

(b) $ab - h^2 \neq 0$ ( )

(c) $a = b$ and $h = 0$ ( )

(d) $a + b = 0$ ( )

7. The shortest distance between the lines

$$\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1} \text{ and } \frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$$

is

(a) $2\sqrt{29}$ ( ) (b) $3\sqrt{29}$ ( )

(c) $2\sqrt{30}$ ( ) (d) $3\sqrt{30}$ ( )

8. The angle between the planes $x+y+z=1$ and $x-y=2$ is

(a) 0 ( )

(b) $\frac{\pi}{2}$ ( )

(c) $\frac{\pi}{3}$ ( )

(d) $\frac{\pi}{4}$ ( )

9. If the sphere is $x^2+y^2+z^2-4x+5y-6z-1=0$, then the radius is

(a) $\frac{7}{2}$ ( )

(b) $\frac{9}{2}$ ( )

(c) $\frac{7}{3}$ ( )

(d) 0 ( )

10. The coordinates of the centre of the circle $x^2+y^2+z^2=30$, $x+2y+3z=14$ is

(a) (1, 2, 3) ( ) (b) (0, 1, 2) ( )

(c) (2, 1, 3) ( ) (d) (4, 3, 1) ( )

(SECTION : B—SHORT ANSWERS)

(*Marks* : 15)

Answer the following : $3\times5=15$

UNIT—I

1. Show that $(\vec{a}\times\vec{b})\times\vec{c}=\vec{a}\times(\vec{b}\times\vec{c})$ if and only if $(\vec{c}\times\vec{a})\times\vec{b}=0$.

OR

2. Find a unit vector perpendicular to each of the vectors $\vec{a}=4\hat{i}-\hat{j}+3\hat{k}$ and $\vec{b}=-2\hat{i}+\hat{j}-2\hat{k}$.

UNIT—II

3. If ϕ and $\vec{A}$ have continuous second-order partial derivatives, then prove that $\text{curl}\,\text{grad}\,\phi=0$.

OR

4. Find the unit normal to the surface $x^4-3xyz+z^2+1=0$ at the point (1, 1, 1).

UNIT—III

5. Find the transformed equation of the curve $(x+2y+4)(2x-y+5)=25$, when the two perpendicular lines $x+2y+4=0$ and $2x-y+5=0$ are taken as coordinate axes.

OR

6. For what value of k, will the equation $3x^2+kxy-3y^2+29x-3y+18=0$ represent a pair of straight lines?

UNIT—IV

7. Show that the lines

$$\frac{x+2}{3}=\frac{1-y}{-1}=\frac{z+1}{-2} \text{ and } \frac{3-x}{-2}=\frac{y}{-2}=\frac{z+2}{1}$$

are coplanar.

OR

8. Find the equations of the planes bisecting the angles between the planes

$$x+2y+2z-3=0 \quad \text{and} \quad 3x+4y+12z+1=0$$

UNIT—V

9. Find the equation of the tangent plane to the sphere

$$x^2+y^2+z^2-2x+3y-4z+6=0$$

at the point (5, 4, 4).

OR

10. Find the equation of the cone whose vertex is (0, 0, 3) and the base is $x^2-y^2=7$.

(**SECTION : C—DESCRIPTIVE**)

(*Marks* : 50)

Answer the following questions : 10×5=50

UNIT—I

1. *(a)* In a triangle ABC on a plane, let $\overrightarrow{BC}=\vec{a}$, $\overrightarrow{CA}=\vec{b}$, $\overrightarrow{AB}=\vec{c}$. Prove that

$$\cos A=\frac{b^2+c^2-a^2}{2bc}$$

where a, b, c are the lengths of the sides BC, CA, AB, respectively. 5

(b) Show that $[\vec{b}\times\vec{c} \quad \vec{c}\times\vec{a} \quad \vec{a}\times\vec{b}]=[\vec{a} \quad \vec{b} \quad \vec{c}]^2$. 5

OR

2. *(a)* If $\vec{r}(t)=(t+1)\hat{i}+2t\hat{j}+t^2\hat{k}$, $t=1$, find the tangential and normal components of acceleration of the given motion and hence write acceleration in the form $\vec{a}=a_T\vec{T}+a_N\vec{N}$ without finding $\vec{T}$ and $\vec{N}$. 5

(b) Prove that for a plane triangle ABC

$$\frac{\sin C}{c} = \frac{\sin B}{b} = \frac{\sin A}{a}$$

where $\overrightarrow{BC} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$, $\overrightarrow{AB} = \vec{c}$. 5

UNIT—II

3. (a) Find an equation for the tangent plane to the surface $xz^2 + x^2y = z - 1$ at the point $(1, -3, 2)$. 5

(b) Let $\vec{B} = 2xz^2\hat{i} - yz\hat{j} + 3xz^3\hat{k}$ and $\phi = x^2yz$. Find the value of $\nabla \times \vec{B}$ and $\operatorname{curl}(\phi\vec{B})$ at the point $(1, 1, 1)$. 5

OR

4. (a) If $\phi = 2xy^2z + x^2y$, evaluate $\int_C \phi \odot d\vec{r}$, where C is the curve $x = t$, $y = t^2$, $z = t^3$ from $t = 0$ to $t = 1$. 5

(b) Evaluate $\iint_S \vec{A} \cdot \vec{n}\, dS$, where $\vec{A} = (x + y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ and S is the surface of the plane $2x + y + 2z = 6$ in the first-octant. 5

UNIT—III

5. (a) Prove that the equation $2x^2 - 5xy + 3y^2 - 2x + 3y = 0$ represents a pair of straight lines. 4

(b) If by any change of axes, without change of origin, the quantity

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c$$

transforms to

$$a'x'^2 + 2h'x'y' + b'y'^2 + 2g'x' + 2f'y' + c'$$

then show that—

(i) $ab - h^2 = a'b' - h'^2$

(ii) $f^2 + g^2 = f'^2 + g'^2$ 6

OR

6. *(a)* Reduce the equation $3x^2 + 2xy + 3y^2 - 16x + 20 = 0$ to standard form. 5

   (b) Find the equation of the diameter of the conic $4x^2 + 6xy - 5y^2 = 1$ conjugate to the diameter $y - 2x = 0$. 5

UNIT—IV

7. *(a)* Prove that the length of the perpendicular drawn from the point (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$ is

$$\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$ 5

   (b) Find the equation of the plane passing through the points $(4, 0, -2)$, $(1, 1, -1)$ and origin. 5

OR

8. *(a)* Find the angle between the planes represented by

$$12x^2 - 2y^2 - 6z^2 - 2xy + 7yz + 6zx = 0$$ 5

   (b) Find the magnitude and the position of the line of shortest distance between the lines

$$2x + y - z = 0 = x - y + 2z \quad \text{and} \quad x + 2y - 3z - 4 = 0 = 2x - 3y + 4z - 5$$ 5

UNIT—V

9. *(a)* Find the equation of the sphere that passes through the circle

$$x^2 + y^2 + z^2 - 2x + 3y - 4z + 6 = 0,\ 3x - 4y + 5z - 15 = 0$$

   and cuts the sphere

$$x^2 + y^2 + z^2 + 2x + 4y - 6z + 11 = 0$$

   orthogonally. 5

(b) Find the equation of the cone, whose vertex is (α, β, γ) and the base is the parabola $z = 0,\ y^2 = 4ax$.

OR

10\. (a) Find the equation of the cylinder whose generators are parallel to the line $2x = y = 3z$ and which passes through the circle $y = 0,\ x^2 + z^2 = 8$.

(b) Find the equation of the right circular cylinder of radius 2, whose axis is the straight line

$$\frac{x}{1} = \frac{y}{-2} = \frac{z}{2}$$ 5

***

 Student's Copy

2024

(CBCS)

(4th Semester)

MATHEMATICS

FOURTH PAPER

(Vector Calculus and Solid Geometry)

Full Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

(SECTION : A—OBJECTIVE)

(*Marks* : 10)

Tick (✓) the correct answer in the brackets provided : 1×10=10

1. The value of $[i\ k\ j]$ is

   (a) 1 ( )

   (b) 2 ( )

   (c) 0 ( )

   (d) −1 ( )

2. If $\vec{a}, \vec{b}, \vec{c}$ are any three vectors, then

$$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b})$$

is

(a) 0 ( )

(b) 1 ( )

(c) −1 ( )

(d) $\vec{b}$ ( )

3. If $\vec{V} = (2x + y)\hat{i} - (3y + 2z)\hat{j} + (x + az)\hat{k}$ is solenoidal, then the value of the constant a is

(a) $2x$ ( )

(b) $2z$ ( )

(c) 1 ( )

(d) 0 ( )

4. If $\phi(x, y, z) = x^2 + y^2 + z^2$, then the value of grad ϕ at the point (1, 2, 3) is

(a) $2\hat{i} + 4\hat{j} + 5\hat{k}$ ( )

(b) $\hat{i} + 4\hat{j} + 6\hat{k}$ ( )

(c) $2\hat{i} + 3\hat{j} + 6\hat{k}$ ( )

(d) $2\hat{i} + 4\hat{j} + 6\hat{k}$ ( )

5. By changing the origin to (2, 3) without changing the direction of axes, the equation $5x + 3y = 3$ changes to

(a) $5x + 3y = 1$ ( )

(b) $5x + 3y + 16 = 0$ ( )

(c) $5x + 3y + 6 = 0$ ( )

(d) None of the above ( )

6. The equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a circle, if

(a) $ab - h^2 = 0$ ( )

(b) $ab - h^2 \neq 0$ ( )

(c) $a = b$ and $h = 0$ ( )

(d) $a + b = 0$ ( )

7. The shortest distance between the lines

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} \text{ and } \frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$$

is

(a) $2\sqrt{29}$ ( ) (b) $3\sqrt{29}$ ( )

(c) $2\sqrt{30}$ ( ) (d) $3\sqrt{30}$ ( )

8. The angle between the planes $x+y+z=1$ and $x-y=2$ is

(a) 0 ( )

(b) $\frac{\pi}{2}$ ( )

(c) $\frac{\pi}{3}$ ( )

(d) $\frac{\pi}{4}$ ( )

9. If the sphere is $x^2+y^2+z^2-4x+5y-6z-1=0$, then the radius is

(a) $\frac{7}{2}$ ( )

(b) $\frac{9}{2}$ ( )

(c) $\frac{7}{3}$ ( )

(d) 0 ( )

10. The coordinates of the centre of the circle $x^2+y^2+z^2=30$, $x+2y+3z=14$ is

(a) (1, 2, 3) ( ) (b) (0, 1, 2) ( )

(c) (2, 1, 3) ( ) (d) (4, 3, 1) ( )

(SECTION : B—SHORT ANSWERS)

(*Marks* : 15)

Answer the following : 3×5=15

UNIT—I

1. Show that $(\vec{a}\times\vec{b})\times\vec{c}=\vec{a}\times(\vec{b}\times\vec{c})$ if and only if $(\vec{c}\times\vec{a})\times\vec{b}=0$.

OR

2. Find a unit vector perpendicular to each of the vectors $\vec{a}=4\hat{i}-\hat{j}+3\hat{k}$ and $\vec{b}=-2\hat{i}+\hat{j}-2\hat{k}$.

UNIT—II

3. If ϕ and $\vec{A}$ have continuous second-order partial derivatives, then prove that $\operatorname{curl}\operatorname{grad}\phi=0$.

OR

4. Find the unit normal to the surface $x^4-3xyz+z^2+1=0$ at the point (1, 1, 1).

UNIT—III

5. Find the transformed equation of the curve $(x+2y+4)(2x-y+5)=25$, when the two perpendicular lines $x+2y+4=0$ and $2x-y+5=0$ are taken as coordinate axes.

OR

6. For what value of k, will the equation $3x^2+kxy-3y^2+29x-3y+18=0$ represent a pair of straight lines?

UNIT—IV

7. Show that the lines

$$\frac{x+2}{3}=\frac{1-y}{-1}=\frac{z+1}{-2} \text{ and } \frac{3-x}{-2}=\frac{y}{-2}=\frac{z+2}{1}$$

are coplanar.

OR

8. Find the equations of the planes bisecting the angles between the planes

$$x + 2y + 2z - 3 = 0 \quad \text{and} \quad 3x + 4y + 12z + 1 = 0$$

UNIT—V

9. Find the equation of the tangent plane to the sphere

$$x^2 + y^2 + z^2 - 2x + 3y - 4z + 6 = 0$$

at the point (5, 4, 4).

OR

10. Find the equation of the cone whose vertex is (0, 0, 3) and the base is $x^2 - y^2 = 7$.

(SECTION : C—DESCRIPTIVE)

(*Marks* : 50)

Answer the following questions : $10 \times 5 = 50$

UNIT—I

1. (a) In a triangle ABC on a plane, let $\overrightarrow{BC} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$, $\overrightarrow{AB} = \vec{c}$. Prove that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

where a, b, c are the lengths of the sides BC, CA, AB, respectively. 5

(b) Show that $[\vec{b} \times \vec{c} \quad \vec{c} \times \vec{a} \quad \vec{a} \times \vec{b}] = [\vec{a} \quad \vec{b} \quad \vec{c}]^2$. 5

OR

2. (a) If $\vec{r}(t) = (t+1)\hat{i} + 2t\hat{j} + t^2\hat{k}$, $t = 1$, find the tangential and normal components of acceleration of the given motion and hence write acceleration in the form $\vec{a} = a_T\vec{T} + a_N\vec{N}$ without finding $\vec{T}$ and $\vec{N}$. 5

(b) Prove that for a plane triangle ABC

$$\frac{\sin C}{c} = \frac{\sin B}{b} = \frac{\sin A}{a}$$

where $\overrightarrow{BC} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$, $\overrightarrow{AB} = \vec{c}$. 5

UNIT—II

3. (a) Find an equation for the tangent plane to the surface $xz^2 + x^2y = z - 1$ at the point $(1, -3, 2)$. 5

(b) Let $\vec{B} = 2xz^2\hat{i} - yz\hat{j} + 3xz^3\hat{k}$ and $\phi = x^2yz$. Find the value of $\nabla \times \vec{B}$ and $\operatorname{curl}(\phi\vec{B})$ at the point $(1, 1, 1)$. 5

OR

4. (a) If $\phi = 2xy^2z + x^2y$, evaluate $\int_C \phi \odot d\vec{r}$, where C is the curve $x = t$, $y = t^2$, $z = t^3$ from $t = 0$ to $t = 1$. 5

(b) Evaluate $\iint_S \vec{A} \cdot \vec{n}\, dS$, where $\vec{A} = (x + y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ and S is the surface of the plane $2x + y + 2z = 6$ in the first-octant. 5

UNIT—III

5. (a) Prove that the equation $2x^2 - 5xy + 3y^2 - 2x + 3y = 0$ represents a pair of straight lines. 4

(b) If by any change of axes, without change of origin, the quantity

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c$$

transforms to

$$a'x'^2 + 2h'x'y' + b'y'^2 + 2g'x' + 2f'y' + c'$$

then show that—

(i) $ab - h^2 = a'b' - h'^2$

(ii) $f^2 + g^2 = f'^2 + g'^2$ 6

OR

6. *(a)* Reduce the equation $3x^2 + 2xy + 3y^2 - 16x + 20 = 0$ to standard form. 5

(b) Find the equation of the diameter of the conic $4x^2 + 6xy - 5y^2 = 1$ conjugate to the diameter $y - 2x = 0$. 5

UNIT—IV

7. *(a)* Prove that the length of the perpendicular drawn from the point (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$ is

$$\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$ 5

(b) Find the equation of the plane passing through the points $(4, 0, -2)$, $(1, 1, -1)$ and origin. 5

OR

8. *(a)* Find the angle between the planes represented by

$$12x^2 - 2y^2 - 6z^2 - 2xy + 7yz + 6zx = 0$$ 5

(b) Find the magnitude and the position of the line of shortest distance between the lines

$$2x + y - z = 0 = x - y + 2z \quad \text{and} \quad x + 2y - 3z - 4 = 0 = 2x - 3y + 4z - 5$$ 5

UNIT—V

9. *(a)* Find the equation of the sphere that passes through the circle

$$x^2 + y^2 + z^2 - 2x + 3y - 4z + 6 = 0, \; 3x - 4y + 5z - 15 = 0$$

and cuts the sphere

$$x^2 + y^2 + z^2 + 2x + 4y - 6z + 11 = 0$$

orthogonally. 5

(b) Find the equation of the cone, whose vertex is (α, β, γ) and the base is the parabola $z = 0$, $y^2 = 4ax$. 5

OR

10. (a) Find the equation of the cylinder whose generators are parallel to the line $2x = y = 3z$ and which passes through the circle $y = 0$, $x^2 + z^2 = 8$. 5

(b) Find the equation of the right circular cylinder of radius 2, whose axis is the straight line

$$\frac{x}{1} = \frac{y}{-2} = \frac{z}{2}$$

5

* * *