**2023**

( NEP—2020 )

( 1st Semester )

## MATHEMATICS (MAJOR/MINOR)

**( Calculus )**

*Full Marks* : 75

*Time* : 3 hours

*The figures in the margin indicate full marks for the questions*

### ( SECTION : A—OBJECTIVE )

( *Marks* : 10 )

Put a Tick (✓) mark against the correct answer in the brackets provided : 1×10=10

**1.** The value of $\lim_{x\to 0}(1+2x)^{\frac{x+3}{x}}$ is

(a) 0 ( )

(b) $e^{\log 2}$ ( )

(c) $e^3$ ( )

(d) $e^6$ ( )

**2.** The differential coefficient of $e^{e^x}$ is

(a) $e^{e^x}\cdot e^x(\log x+1)$ ( )

(b) $e^{e^x}\cdot e^x$ ( )

(c) $e^x(x+1)$ ( )

(d) $xe^{e^x}$ ( )

3. The $n$th derivative of $y = \frac{1}{a-x}$ is

(a) $\frac{n!}{(a-x)^{n+1}}$ ( )

(b) $\frac{(-1)^n}{n!(a-x)^n}$ ( )

(c) $\frac{n!}{(x-a)^n}$ ( )

(d) $\frac{(-1)n!}{(a-x)^{n+1}}$ ( )

4. The value of $c$ in Lagrange's mean value theorem for the function $f(x) = e^x$ in $[0,1]$ is

(a) $\log(e-1)$ ( )

(b) $\log(e+1)$ ( )

(c) $\log(e-2)$ ( )

(d) $\log(e+2)$ ( )

5. $f(x) = e^x$ can be expressed in power of $(x+3)$ by using

(a) Maclaurin's theorem ( )

(b) Leibnitz's theorem ( )

(c) Taylor's theorem ( )

(d) None of these ( )

6. The value of $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$ is

(a) 0 ( )

(b) $\frac{\pi}{4}$ ( )

(c) $\frac{\pi}{2}$ ( )

(d) None of these ( )

7. The value of $\int_0^{\pi/2} \sin^4 x \cos^5 x \, dx$ is

(a) $\frac{8}{315}$ ( )

(b) $-\frac{8}{315}$ ( )

(c) $\frac{315}{8}$ ( )

(d) $-\frac{315}{8}$ ( )

8. The value of $\int_0^{\pi/4} \tan^5 \theta \, d\theta$ is

(a) $\frac{1}{4}(\log 2 - 1)$ ( )

(b) $\frac{1}{4}(2\log 2 - 1)$ ( )

(c) $\frac{1}{4}\log 2$ ( )

(d) None of the above ( )

9. If $u(x, y)$ is a homogeneous function of degree $n$, then the value of

$$x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2}$$

is

(a) $n(n-1)u$ ( )

(b) $(n-1)\frac{\partial u}{\partial y}$ ( )

(c) $nu$ ( )

(d) $(n-1)\frac{\partial u}{\partial x}$ ( )

10. Every Cauchy sequence must be

(a) monotonic ( )

(b) bounded above only ( )

(c) bounded below only ( )

(d) bounded ( )

## ( SECTION : B—SHORT ANSWERS )

( *Marks* : 15 )

Answer *five* questions, taking at least *one* from each Unit : 3×5=15

### UNIT—I

1. If the area of a circle increases at a uniform rate, then show that the rate of increase of the perimeter varies inversely as the radius.

2. Differentiate $x^{\sin^{-1} x}$ with respect to $\sin^{-1} x$.

### UNIT—II

3. Let $f(x) = A + Bx + Cx^2$ in $[a, b]$. Show that the value of $c$ in the mean value theorem

$$f'(c) = \frac{f(b) - f(a)}{b - a} \text{ is } \frac{a + b}{2}$$

4. If $f(x) = (x - a)^m (x - b)^n$, where $m$ and $n$ are positive integers, then show that $c$ in Rolle's theorem divides the segment $a \le x \le b$ in the ratio $m : n$.

### UNIT—III

5. Show that

$$\int_0^1 \frac{\log(1 + x)}{1 + x^2} dx = \frac{\pi}{8} \log 2$$

6. Obtain the reduction formula for

$$\int_0^1 x^m e^x dx$$

## UNIT—IV

**7.** If $u = \tan^{-1}\dfrac{x^2+y^2}{x-y}$, then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$$

**8.** Test the convergency of the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{\sqrt{n^2+1}} x^n, \; x > 0$$

## ( SECTION : C—DESCRIPTIVE )

( *Marks* : 50 )

Answer *five* questions, taking at least *one* from each Unit : 10×5=50

## UNIT—I

**1.** *(a)* Draw the graph of the function defined by

$$f(x) = \begin{cases} x^2, & \text{when } x < 0 \\ x, & \text{when } 0 \le x \le 1 \\ \dfrac{1}{x}, & \text{when } x > 1 \end{cases}$$

Discuss whether $f(x)$ is continuous at $x = 1$. 6

*(b)* Evaluate the following using L'Hospital rule : 4

$$\lim_{x \to 0} \frac{e^x + e^{-x} - 2\cos x}{x \sin x}$$

**2.** *(a)* Using ε-δ definition of continuity, prove that

$$f(x) = \begin{cases} x^2 \cos\left(\dfrac{1}{x}\right), & \text{when } x \ne 0 \\ 0, & \text{when } x = 0 \end{cases}$$

is continuous at $x = 0$. 4

(b) If $y = e^{a\sin^{-1} x}$, then show that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$$

Hence evaluate $(y_n)_0$. 6

UNIT—II

3. (a) State and prove Rolle's theorem. 5

(b) Prove that

$$\sin ax = ax - \frac{a^3x^3}{3!} + \frac{a^5x^5}{5!} - \cdots + \frac{a^{n-1}x^{n-1}}{(n-1)!}\sin\left(\frac{n-1}{2}\pi\right) + \frac{a^nx^n}{n!}\sin\left(a\theta x + \frac{n\pi}{2}\right)$$

where $0 < \theta < 1$. 5

4. (a) State and prove Taylor's theorem. 6

(b) Let $f(x)$ be defined and continuous on $[a-h, a+h]$ and derivable on $]a-h, a+h[$. Prove that there is a real number $\theta$ between 0 and 1 for which

$$f(a+h) - f(a-h) = h\{f'(a+\theta h) - f'(a-\theta h)\}$$ 4

UNIT—III

5. (a) If $I_n = \int_0^a (a^2-x^2)^n dx$, $n > 0$, then prove that $(2n+1)I_n = 2na^2 I_{n-1}$. Hence evaluate

$$\int_0^a (a^2-x^2)^3 dx$$ 5

(b) Use the definition of definite integral as a limit of sum to evaluate

$$\int_1^4 \frac{1}{x} dx$$ 5

6. (a) If $I_n = \int_0^{\pi/2} x^n \sin x\, dx$, $n$ be a positive integer greater than 1, then obtain the reduction formula for $I_n$ and hence evaluate

$$\int_0^{\pi/2} x^5 \sin x\, dx$$ 6

(b) If $I_n = \int_0^{\pi/4} \tan^n x\, dx$, then prove that $n(I_{n+1} + I_{n-1}) = 1$. 4

## UNIT—IV

7. (a) If

$$u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$$

then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$$ 5

(b) Show that the sequence $\{S_n\}_{n=1}^{\infty}$, where

$$S_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}, \ \forall n \in \mathbb{N}$$

is convergent. 5

8. (a) If $u = \log\sqrt{x^2+y^2+z^2}$, then prove that

$$(x^2+y^2+z^2)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 1$$ 5

(b) Examine the convergence for the series

$$\sum_{n=0}^{\infty}\left(\frac{n+1}{n+2}\right)x^n$$ 5

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