MATH100 (MAJOR) **Student's Copy**

2 0 2 3

(NEP—2020)

(1st Semester)

MATHEMATICS (MAJOR)

(Vector Analysis)

Full Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

(SECTION : A—OBJECTIVE)

(*Marks* : 10)

Tick (✓) the correct answer in the brackets provided : 1×10=10

1. The unit tangent vector to the space curve $x = t^2 + 2$, $y = 4t - 5$, $z = 2t^2 - 6t$ at $t = 2$ is

   (a) $\frac{1}{\sqrt{3}}(-2\hat{i} + 2\hat{j} + \hat{k})$ ( )

   (b) $-\frac{1}{\sqrt{3}}(-2\hat{i} + 2\hat{j} + \hat{k})$ ( )

   (c) $-\frac{1}{\sqrt{3}}(-2\hat{i} - 2\hat{j} + \hat{k})$ ( )

   (d) $\frac{1}{\sqrt{3}}(2\hat{i} + 2\hat{j} + \hat{k})$ ( )

2. Which of the following is correct?

(a) $\vec{u}\cdot\frac{d\vec{u}}{dt}=|\vec{u}|\frac{d|\vec{u}|}{dt}$ ( )

(b) $\vec{u}\cdot\frac{d\vec{u}}{dt}=2|\vec{u}|\frac{d|\vec{u}|}{dt}$ ( )

(c) $2\vec{u}\cdot\frac{d\vec{u}}{dt}=|\vec{u}|\frac{d|\vec{u}|}{dt}$ ( )

(d) $-\vec{u}\cdot\frac{d\vec{u}}{dt}=|\vec{u}|\frac{d|\vec{u}|}{dt}$ ( )

3. For any vector $\vec{a}$, div $(\text{curl}\,\vec{a})=$

(a) 3 ( )
(b) 2 ( )
(c) 1 ( )
(d) 0 ( )

4. If $\vec{a}$ is any vector and $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, then $(\vec{a}\cdot\nabla)\vec{r}$ is equal to

(a) $\vec{r}$ ( )
(b) $\vec{a}$ ( )
(c) $\vec{a}\times\vec{r}$ ( )
(d) $\vec{a}\cdot\vec{r}$ ( )

5. If the vector $\vec{V}=y^2z\,\hat{i}+axyz\,\hat{j}+xy^2\hat{k}$ be a conservative vector, then a is equal to

(a) 0 ( )
(b) 2 ( )
(c) 1 ( )
(d) 3 ( )

6. The line integral of the vector function $\vec{u}(x, y)=2y\hat{i}+x\hat{j}$ along the straight line from (0, 0) to (2, 4) is

(a) 0 ( )
(b) 10 ( )
(c) 11 ( )
(d) 12 ( )

7. For a given vector $\vec{V}$, the projection R of a surface S on the zx-plane is

(a) $\iint\limits_S \vec{A}\cdot\hat{n}\,dS = \iint\limits_R \vec{A}\cdot\hat{n}\dfrac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$ ( )

(b) $\iint\limits_S \vec{A}\cdot\hat{n}\,dS = \iint\limits_R \vec{A}\cdot\hat{n}\dfrac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$ ( )

(c) $\iint\limits_S \vec{A}\cdot\hat{n}\,dS = \iint\limits_R \vec{A}\cdot\hat{n}\dfrac{dz\,dx}{|\hat{n}\cdot\hat{j}|}$ ( )

(d) $\iint\limits_S \vec{A}\cdot\hat{n}\,dS = \iint\limits_R \vec{A}\cdot\hat{n}\dfrac{dx\,dz}{|\hat{n}\cdot\hat{i}|}$ ( )

8. If V is the volume enclosed by the surface S, then the value of $\iint\limits_S \vec{r}\cdot\hat{n}dS$ is

(a) V ( )

(b) 2V ( )

(c) 3V ( )

(d) 4V ( )

9. Stokes' theorem is the relation between

(a) line and surface integrals ( )

(b) surface and volume integrals ( )

(c) line and volume integrals ( )

(d) line, surface and volume integrals ( )

10. Which one of the following theorems uses curl operation?

(a) Green's theorem ( )

(b) Gauss' divergence theorem ( )

(c) Ampere's law ( )

(d) Stokes' theorem ( )

(SECTION : B—SHORT ANSWERS)

(*Marks* : 15)

Answer *five* questions, taking at least *one* from each Unit : $3 \times 5 = 1$

UNIT—I

1. Prove that

$$\frac{d}{dt}[\vec{p}\,\vec{q}\,\vec{r}] = \left[\frac{d\vec{p}}{dt}\vec{q}\,\vec{r}\right] + \left[\vec{p}\frac{d\vec{q}}{dt}\vec{r}\right] + \left[\vec{p}\,\vec{q}\frac{d\vec{r}}{dt}\right]$$

where $\vec{p}\,\vec{q}\,\vec{r}$ are functions of t. Hence find

$$\frac{d}{dt}\left[\vec{r}\frac{d\vec{r}}{dt}\frac{d^2\vec{r}}{dt^2}\right]$$

2. A particle moves along the curves $x = 2t^2$, $y = (t^2 - 4t)$, $z = (3t - 5)$, where t is the time. Find the velocity and acceleration at $t = 1$. Also, find the direction of motion at the given value of t.

UNIT—II

3. Find the value of curl (grad ϕ), where ϕ is a scalar function.

4. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1, -2, -1)$ in the direction $2\hat{i} - \hat{j} - 2\hat{k}$. Also, find the magnitude of greatest directional derivative at the same point.

UNIT—III

5. If $\vec{F} = 3xy\hat{i} - y^2\hat{j}$, evaluate $\int_C \vec{F} \cdot d\vec{r}$, where C is the curve in the *xy*-plane, $y = 2x^2$ from $(0, 0)$ to $(1, 2)$.

6. Find the work done when a force $\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$ moves a particle from $(0, 0)$ to $(1, 1)$ along a parabola $y^2 = x$.

UNIT—IV

7. Evaluate $\oint_C (yz\,dx + zx\,dy + xy\,dz)$ by Stokes' theorem, where C is the curve $x^2 + y^2 = 1,\ z = y^2$.

8. Prove that

$$\iiint_V \frac{dV}{r^2} = \iint_S \frac{\vec{r}\cdot\vec{n}}{r^2}\,dS$$

(SECTION : C—DESCRIPTIVE)

(*Marks* : 50)

Answer *five* questions, taking at least *one* from each Unit : 10×5=50

UNIT—I

1. *(a)* If $\vec{r}\,' = \dfrac{d\vec{r}}{dt},\ \vec{r}\,'' = \dfrac{d^2\vec{r}}{dt^2}$, then find the value of $\dfrac{d^2}{dt^2}[\vec{r}\times(\vec{r}\,'\times\vec{r}\,'')]$. 4

(b) Prove that

$$\frac{d}{dt}(\vec{a}\times\vec{b}) = \vec{a}\times\frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt}\times\vec{b}$$

from the definition of derivative of a function. 3

(c) Find the arc length of the indicated portion of the curve

$$\vec{r}(t) = (\cos^3 t)\hat{j} + (\sin^3 t)\hat{k},\ 0 \le t \le \frac{\pi}{2}$$ 3

2. *(a)* Prove that a necessary and sufficient condition that a proper vector $\vec{u}$ remains parallel to a fixed line is that $\vec{u}\times\dfrac{d\vec{u}}{dt} = \vec{0}$. 5

(b) Find the curvature and unit normal vector for the helix

$$\vec{r}(t) = (a\cos t)\hat{i} + (a\sin t)\hat{j} + bt\hat{k},\ a, b \ge 0,\ a^2 + b^2 \ne 0$$ 5

UNIT—II

3. (a) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $r = |\vec{r}|$ and $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ is a constant vector, then find $\operatorname{div}\left(\frac{\vec{a} \times \vec{r}}{r^n}\right)$.

(b) Find the value of n for which the vector $r^n \vec{r}$ is solenoidal where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

4. (a) Show that $\operatorname{div}(\operatorname{grad} r^n) = n(n+1)r^{n-2}$, where $r = \sqrt{x^2 + y^2 + z^2}$. Hence show that $\nabla^2\left(\frac{1}{r}\right) = 0$. 5+1=

(b) Prove that $\operatorname{curl}(\vec{a} \cdot \vec{b}) = \vec{a} \operatorname{curl} \vec{b} + \vec{b} \operatorname{curl} \vec{a} + (\vec{a} \cdot \nabla)\vec{b} + (\vec{b} \cdot \nabla)\vec{a}$.

UNIT—III

5. (a) Let $\vec{F} = 2y\hat{i} - z\hat{j} + x^2\hat{k}$ and S is the surface of the cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$ and $z = 6$. Then evaluate $\iint_S \vec{F} \cdot \vec{n}\, dS$. 5

(b) If $\vec{F} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$, evaluate $\iiint_V \nabla \cdot \vec{F} dV$, where V is the closed region bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + 2y + z = 4$. 5

6. (a) If $\vec{F} = 2z\hat{i} - x\hat{j} + y\hat{k}$, evaluate $\iiint_V \vec{F} dV$, where V is the region bounded by the surfaces $x = 0$, $y = 0$, $x = 2$, $y = 4$, $z = x^2$, $z = 2$. 5

(b) Evaluate $\iint_S \vec{A} \cdot \vec{n}\, dS$, where $\vec{A} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = 1$ which lies in the first octant. 5

UNIT—IV

7. (a) Verify Stokes' theorem for $\vec{A} = (2x - y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$, where S is the upper surface of the sphere $x^2 + y^2 + z^2 = 1$ bounded by its projection on xy-plane. 5

   (b) State Green's theorem and hence apply it to evaluate $\int_C [(2x^2 - y^2)\,dx + (x^2 + y^2)\,dy]$, where C is the boundary of the area enclosed by the x-axis and the upper half of the circle $x^2 + y^2 = a^2$. 5

8. (a) State Gauss divergence theorem. Use Gauss divergence theorem to show that $\iint_S \nabla(x^2 + y^2 + z^2)\,d\vec{S} = 6V$, where V is the volume enclosed by the surface S. 5

   (b) Verify Stokes' theorem for $\vec{F} = (x^2 - y^2)\hat{i} + 2x\hat{j}$, around the rectangle bounded by the straight lines $x = 0$, $x = a$, $y = 0$, $y = b$. 5

★★★

2023

(NEP—(UG))

(1st Semester)

MATHEMATICS (MAJOR)

(Vector Analysis)

Full Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

(SECTION : A—OBJECTIVE)

(Marks : 10)

Tick (✓) the correct answer in the brackets provided : 1×10=10

1. The unit tangent vector to the space curve $x = t^2 + 2$, $y = 4t - 5$, $z = 2t^2 - 6t$ at $t = 2$ is

   (a) [illegible] ( )

   (b) [illegible] ( )

   (c) [illegible] ( )

   (d) [illegible] ( )

2. Which of the following is correct?

(a) $\vec{u} \cdot \frac{d\vec{u}}{dt} = |\vec{u}| \frac{d|\vec{u}|}{dt}$ ( )

(b) $\vec{u} \cdot \frac{d\vec{u}}{dt} = 2|\vec{u}| \frac{d|\vec{u}|}{dt}$ ( )

(c) $2\vec{u} \cdot \frac{d\vec{u}}{dt} = |\vec{u}| \frac{d|\vec{u}|}{dt}$ ( )

(d) $-\vec{u} \cdot \frac{d\vec{u}}{dt} = |\vec{u}| \frac{d|\vec{u}|}{dt}$ ( )

3. For any vector $\vec{a}$, div (curl $\vec{a}$) =

(a) 3 ( ) (b) 2 ( )

(c) 1 ( ) (d) 0 ( )

4. If $\vec{a}$ is any vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then $(\vec{a} \cdot \nabla)\vec{r}$ is equal to

(a) $\vec{r}$ ( ) (b) $\vec{a}$ ( )

(c) $\vec{a} \times \vec{r}$ ( ) (d) $\vec{a} \cdot \vec{r}$ ( )

5. If the vector $\vec{V} = y^2 z\hat{i} + axyz\hat{j} + xy^2\hat{k}$ be a conservative vector, then a is equal to

(a) 0 ( ) (b) 2 ( )

(c) 1 ( ) (d) 3 ( )

6. The line integral of the vector function $\vec{u}(x, y) = 2y\hat{i} + x\hat{j}$ along the straight line from (0, 0) to (2, 4) is

(a) 0 ( ) (b) 10 ( )

(c) 11 ( ) (d) 12 ( )

7. For a given vector $\vec{V}$, the projection R of a surface S on the zx-plane is

(a) $\iint\limits_S \vec{A}\cdot\hat{n}\,dS = \iint\limits_R \vec{A}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$ ( )

(b) $\iint\limits_S \vec{A}\cdot\hat{n}\,dS = \iint\limits_R \vec{A}\cdot\hat{n}\,\frac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$ ( )

(c) $\iint\limits_S \vec{A}\cdot\hat{n}\,dS = \iint\limits_R \vec{A}\cdot\hat{n}\,\frac{dz\,dx}{|\hat{n}\cdot\hat{j}|}$ ( )

(d) $\iint\limits_S \vec{A}\cdot\hat{n}\,dS = \iint\limits_R \vec{A}\cdot\hat{n}\,\frac{dx\,dz}{|\hat{n}\cdot\hat{i}|}$ ( )

8. If V is the volume enclosed by the surface S, then the value of $\iint\limits_S \vec{r}\cdot\hat{n}dS$ is

(a) V ( )

(b) 2V ( )

(c) 3V ( )

(d) 4V ( )

9. Stokes' theorem is the relation between

(a) line and surface integrals ( )

(b) surface and volume integrals ( )

(c) line and volume integrals ( )

(d) line, surface and volume integrals ( )

10. Which one of the following theorems uses curl operation?

(a) Green's theorem ( )

(b) Gauss' divergence theorem ( )

(c) Ampere's law ( )

(d) Stokes' theorem ( )

(SECTION : B—SHORT ANSWERS)

(*Marks* : 15)

Answer *five* questions, taking at least *one* from each Unit : 3×5=15

UNIT—I

1. Prove that

$$\frac{d}{dt}[\vec{p}\,\vec{q}\,\vec{r}] = \left[\frac{d\vec{p}}{dt}\vec{q}\,\vec{r}\right] + \left[\vec{p}\frac{d\vec{q}}{dt}\vec{r}\right] + \left[\vec{p}\,\vec{q}\frac{d\vec{r}}{dt}\right]$$

where $\vec{p}\,\vec{q}\,\vec{r}$ are functions of t. Hence find

$$\frac{d}{dt}\left[\vec{r}\frac{d\vec{r}}{dt}\frac{d^2\vec{r}}{dt^2}\right]$$

2. A particle moves along the curves $x = 2t^2$, $y = (t^2 - 4t)$, $z = (3t - 5)$, where t is the time. Find the velocity and acceleration at $t = 1$. Also, find the direction of motion at the given value of t.

UNIT—II

3. Find the value of curl (grad ϕ), where ϕ is a scalar function.

4. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1, -2, -1)$ in the direction $2\hat{i} - \hat{j} - 2\hat{k}$. Also, find the magnitude of greatest directional derivative at the same point.

UNIT—III

5. If $\vec{F} = 3xy\hat{i} - y^2\hat{j}$, evaluate $\int_C \vec{F} \cdot d\vec{r}$, where C is the curve in the xy-plane, $y = 2x^2$ from $(0, 0)$ to $(1, 2)$.

6. Find the work done when a force $\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$ moves a particle from $(0, 0)$ to $(1, 1)$ along a parabola $y^2 = x$.

UNIT—IV

. Evaluate $\oint_C (yz\,dx + zx\,dy + xy\,dz)$ by Stokes' theorem, where C is the curve $x^2 + y^2 = 1,\ z = y^2$.

. Prove that

$$\iiint_V \frac{dV}{r^2} = \iint_S \frac{\vec{r}\cdot\vec{n}}{r^2}\,dS$$

(SECTION : C—DESCRIPTIVE)

(*Marks* : 50)

nswer *five* questions, taking at least *one* from each Unit : 10×5=50

UNIT—I

1. (a) If $\vec{r}\,' = \dfrac{d\vec{r}}{dt},\ \vec{r}\,'' = \dfrac{d^2\vec{r}}{dt^2}$, then find the value of $\dfrac{d^2}{dt^2}[\vec{r}\times(\vec{r}\,'\times\vec{r}\,'')]$. 4

(b) Prove that

$$\frac{d}{dt}(\vec{a}\times\vec{b}) = \vec{a}\times\frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt}\times\vec{b}$$

from the definition of derivative of a function. 3

(c) Find the arc length of the indicated portion of the curve

$$\vec{r}(t) = (\cos^3 t)\hat{j} + (\sin^3 t)\hat{k},\ 0 \le t \le \frac{\pi}{2}$$ 3

2. (a) Prove that a necessary and sufficient condition that a proper vector $\vec{u}$ remains parallel to a fixed line is that $\vec{u}\times\dfrac{d\vec{u}}{dt} = \vec{0}$. 5

(b) Find the curvature and unit normal vector for the helix

$$\vec{r}(t) = (a\cos t)\hat{i} + (a\sin t)\hat{j} + bt\hat{k},\ a, b \ge 0,\ a^2 + b^2 \ne 0$$ 5

UNIT—II

3. (a) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $r = |\vec{r}|$ and $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ is a constant vector, then find $\text{div}\left(\frac{\vec{a} \times \vec{r}}{r^n}\right)$.

   (b) Find the value of n for which the vector $r^n \vec{r}$ is solenoidal where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. 5

4. (a) Show that $\text{div}(\text{grad } r^n) = n(n+1)r^{n-2}$, where $r = \sqrt{x^2 + y^2 + z^2}$. Hence show that $\nabla^2\left(\frac{1}{r}\right) = 0$. 5+1=6

   (b) Prove that $\text{curl}(\vec{a} \cdot \vec{b}) = \vec{a} \text{ curl } \vec{b} + \vec{b} \text{ curl } \vec{a} + (\vec{a} \cdot \nabla)\vec{b} + (\vec{b} \cdot \nabla)\vec{a}$. 4

UNIT—III

5. (a) Let $\vec{F} = 2y\hat{i} - z\hat{j} + x^2\hat{k}$ and S is the surface of the cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$ and $z = 6$. Then evaluate $\iint_S \vec{F} \cdot \vec{n}\, dS$. 5

   (b) If $\vec{F} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$, evaluate $\iiint_V \nabla \cdot \vec{F} dV$, where V is the closed region bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + 2y + z = 4$. 5

6. (a) If $\vec{F} = 2z\hat{i} - x\hat{j} + y\hat{k}$, evaluate $\iiint_V \vec{F} dV$, where V is the region bounded by the surfaces $x = 0$, $y = 0$, $x = 2$, $y = 4$, $z = x^2$, $z = 2$. 5

   (b) **Evaluate** $\iint_S \vec{A} \cdot \vec{n}\, dS$, where $\vec{A} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ and S is the surface of the **sphere** $x^2 + y^2 + z^2 = 1$ which lies in the first octant. 5

UNIT—IV

(a) Verify Stokes' theorem for $\vec{A} = (2x - y)\hat{i} - yz^2\hat{j} - y^2 z\hat{k}$, where S is the upper surface of the sphere $x^2 + y^2 + z^2 = 1$ bounded by its projection on xy-plane. 5

(b) State Green's theorem and hence apply it to evaluate $\int_C [(2x^2 - y^2)\,dx + (x^2 + y^2)\,dy]$, where C is the boundary of the area enclosed by the x-axis and the upper half of the circle $x^2 + y^2 = a^2$. 5

(a) State Gauss divergence theorem. Use Gauss divergence theorem to show that $\iint_S \nabla(x^2 + y^2 + z^2)\,d\vec{S} = 6V$, where V is the volume enclosed by the surface S. 5

(b) Verify Stokes' theorem for $\vec{F} = (x^2 - y^2)\hat{i} + 2x\hat{j}$, around the rectangle bounded by the straight lines $x = 0$, $x = a$, $y = 0$, $y = b$. 5

★★★