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(CBCS)

(5th Semester)

PHYSICS

FIFTH PAPER

(Mathematical Physics—II)

Full Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

(SECTION : A—OBJECTIVE)

(*Marks* : 10)

Tick (✓) the correct answer in the brackets provided : $1\times10=10$

1. The function $\dfrac{1}{(z-1)^{1/2}}$

   (a) is analytic in the region $|z|<2$ ( )

   (b) has a pole at $z=1$ ( )

   (c) has a branch point at $z=1$ ( )

   (d) has an essential singularity at $z=1$ ( )

2. The value of $\left|\int_C \dfrac{dz}{z}\right|$, where C is $|z|=1$, is equal to

   (a) $2\pi i$ ( ) (b) 2π ( )

   (c) π ( ) (d) 0 ( )

3. For the linear differential equation

$$y'' + \frac{1}{2x-1} y' + \frac{x}{(2x-1)^2} y = 0; \text{ where } y' = \frac{dy}{dx}$$

(a) $x = 0$ is a singular point ( )

(b) $x = 1/2$ is an irregular singular point ( )

(c) $x = 1/2$ is a regular singular point ( )

(d) $x = 1/2$ is an ordinary point ( )

4. If $u(x, t) = \sum_{n=1}^{\infty} \{C_n \cos n\pi ct + D_n \sin n\pi ct\} \sin n\pi x$ be the expression for displacement of a vibrating string at any time t stretched between two fixed points $(0, 0)$ and $(1, 0)$ and released from the position $u(x, 0) = \lambda$ with initial velocity $g(x) = 1$, then

(a) $D_n = \frac{2}{n^2\pi^2 c}(1 - \sin n\pi)$ ( )

(b) $D_n = \frac{2}{n^2\pi^2 c}(1 - \cos n\pi)$ ( )

(c) $D_n = \frac{2}{n^2\pi^2 c}(1 + \sin n\pi)$ ( )

(d) $D_n = \frac{2}{n^2\pi^2 c}(1 + \cos n\pi)$ ( )

5. The value of the integral

$$\int_{-1}^{+1} (P_0 + 2P_1 + 3P_2) P_2 \, dx$$

is equal to

(a) $\frac{6}{5}$ ( )

(b) $\frac{4}{5}$ ( )

(c) 3 ( )

(d) 0 ( )

6. The value of the integral $\int \frac{J_1(x)}{x} dx$ is

(a) $xJ_1(x)+c$ ( )

(b) $\frac{1}{x}J_1(x)+c$ ( )

(c) $-x J_1(x)+c$ ( )

(d) $-\frac{1}{x}J_1(x)+c$ ( )

7. The finite Fourier sine transform of $\frac{x}{\pi}$ in the interval $(0, \pi)$ is

(a) $\frac{(-1)^{n+1}}{n}$ ( )

(b) $\frac{(-1)^n}{n}$ ( )

(c) $(-1)^n$ ( )

(d) $(-1)^{n+1}$ ( )

8. The Fourier transform of $\frac{df}{dt}$, i.e., FT $\left[\frac{df}{dt}\right]$ is

(a) $\frac{\omega}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$ ( )

(b) $\sqrt{\frac{\omega}{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i\omega t}dt$ ( )

(c) $\frac{i\omega}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$ ( )

(d) $\frac{1}{i\omega\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i\omega t}dt$ ( )

9. If $\mathscr{L}[\sin t]=\frac{1}{s^2+1}$, then $\int_0^{\infty}\frac{\sin t}{t}dt$ is equal to

(a) 0 ( )

(b) $\frac{\pi}{4}$ ( )

(c) $\frac{\pi}{2}$ ( )

(d) π ( )

10. If $f(s)$ is the Laplace transform of $F(t)$, then $\mathscr{L}^{-1}[f(s \pm a)]$ is

(a) $e^{\pm at}F(t)$ ( )

(b) $e^{\mp at}F(t)$ ( )

(c) $e^{\pm at}F(at)$ ( )

(d) $e^{\mp at}F(at)$ ( )

(SECTION : B—SHORT ANSWERS)

(*Marks* : 15)

Answer the following questions in brief : $3\times5=15$

UNIT—I

1. Find the analytic function $f(z)$, whose real part is $e^x \cos y$.

OR

2. Evaluate the integral

$$\oint_C \frac{dz}{z-\frac{3}{2}}$$

where C is a circle $|z-1|=1$.

UNIT—II

3. Find the regular singular points of the differential equation

$$x^2(x-2)^3 y'' + 2(x-2)y' + (x+3)y = 0$$

OR

4. Obtain the solution of indicial equation for the differential equation

$$y'' + \frac{3}{x}y' + \frac{(3-x^2)}{x^2}y = 0$$

UNIT—III

5. Show that $xJ_n'(x) = -nJ_n(x) + xJ_{n-1}(x)$, where $J_n(x)$ is Bessel's function.

OR

6. Using Rodrigue's formula for $P_n(x)$, prove that

$$\int_{-1}^{+1} P_n(x)dx = 0; \qquad (n \neq 0)$$

UNIT—IV

7. For a half-wave rectifier, current is given by

$$I = \begin{cases} I_0 \sin \omega t & ; \quad 0 \le t \le T/2 \\ 0 & ; \quad T/2 \le t \le T \end{cases}$$

Show that the Fourier coefficient $b_n = 0$ for all values of n.

OR

8. If $g(\omega)$ is the Fourier transform of $f(t)$, then show that the Fourier transform of $f(t)\cos at$ is $\frac{1}{2}[g(\omega-a)+g(\omega+a)]$.

UNIT—V

9. Evaluate the inverse Laplace transform of

$$\frac{1}{s^2(s^2-\omega^2)}$$

OR

10. Show that the Laplace transform of $e^t \cos \omega t$ is

$$\frac{s-1}{(s-1)^2+\omega^2}$$

(SECTION : C—DESCRIPTIVE)

(*Marks* : 50)

Answer the following questions : 10×5=50

UNIT—I

1. (a) Derive the polar form of Cauchy-Riemann equation for the analyticity of a complex function. 5

(b) Using Cauchy's residue theorem, evaluate

$$\frac{1}{2\pi i}\int_C \frac{e^{zt}}{z^2(z^2-2z+2)}dz$$

where C is the circle $|z|=3$. 5

OR

2. (a) If $f(z)$ is analytic inside and on a simple closed curve C and a is any point inside C, then show that

$$f(a)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$$ 6

(b) Expand $f(z)=\frac{1}{(z+1)(z+3)}$ in a Laurent series valid for $1<|z|<3$. 4

UNIT—II

3. *(a)* Obtain the power series solution of the differential equation

$$\frac{d^2y}{dz^2} - \frac{2z}{(1-z^2)}\frac{dy}{dz} + \frac{2}{(1-z^2)}y = 0$$

about $z = 1$. 7

(b) Using the method of separation of variables, solve the differential equation

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$ 3

OR

4. *(a)* Write down the Laplace equation in 2-D polar coordinates and solve it by the method of separation of variables. 1+5=6

(b) If $u(x, t) = \sum_{n=1}^{\infty} C_n \cos n\pi ct \sin n\pi x$ be the expression for subsequent displacement of a string stretched between two fixed points and released at rest from the initial position $u(x, 0) = \lambda \sin \pi x$, show that $C_n = 0$ for $n > 1$. 4

UNIT—III

5. *(a)* Show that $H_n(x)$ is the coefficient of z^n in the expansion of $e^{x^2-(z-x)^2}$ in ascending powers of z. Hence prove that $H_n'(x) = 2n H_{n-1}(x)$. 4+3=7

(b) Using the recursion relation $nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$, show that

$$\int_{-1}^{+1} xP_n(x)P_{n-1}(x)\,dx = \frac{2n}{4n^2-1}$$ 3

OR

6. (a) Starting from the expression

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

for Bessel's function, prove the following : 3+3=6

(i) $\frac{d}{dx}[J_0(x)] = -J_1(x)$

(ii) $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

(b) Using the expression $e^{x(z-\frac{1}{z})/2} = \sum_{-\infty}^{+\infty} z^n J_n(x)$, show that

(i) $\cos(x \sin\varphi) = J_0(x) + 2\cos 2\varphi\, J_2(x) + 2\cos 4\varphi\, J_4(x) + \cdots$

(ii) $\sin(x \sin\varphi) = 2\sin\varphi\, J_1(x) + 2\sin 3\varphi\, J_3(x) + 2\sin 5\varphi\, J_5(x) + \cdots$ 2+2=4

UNIT—IV

7. (a) Express the function

$$f(x) = \begin{cases} 1 & \text{for } |x| \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

as a Fourier integral. 4

(b) Find the Fourier sine transform of $\frac{e^{-ax}}{x}$. 4

(c) If $g(\omega)$ is the Fourier transform of $f(t)$, show that the Fourier transform of $f(at)$ is $\frac{1}{a} g\left(\frac{\omega}{a}\right)$. 2

OR

8. (a) Obtain the Fourier series of a function $f(x) = x^2$; $-\pi \le x \le \pi$. Hence show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

4+1=5

(b) Find the Fourier transform of $e^{-|t|}$. 2

(c) Using the properties of Dirac delta function, prove that

$$\delta(ax) = \frac{1}{a}\delta(x)\,; \quad a > 0$$

3

UNIT—V

9. (a) If $F(t)$ of period T such that $F(t+nT) = F(t)$, show that

$$\mathscr{L}[F(t)] = \frac{1}{(1-e^{-sT})}\int_0^T e^{-st}F(t)dt$$

Hence find the Laplace transform of sawtooth wave function $F(t) = \frac{at}{T}$ for $0 < t < T$.

4

(b) Apply residue method to find the inverse Laplace transform of $\frac{a}{s^2 - a^2}$. 3

(c) Using Laplace transform, show that

$$\int_0^\infty t^2 e^{-t}\sin t\,dt = \frac{1}{2}$$

3

OR

10. (a) Show that

$$\mathscr{L}[S_i(t)] = \frac{1}{s}\tan^{-1}\left(\frac{1}{s}\right)$$

where sine integral function $S_i(t) = \int_0^t \frac{\sin x}{x}dx$.

4

(b) Use the Laplace transform method to solve the differential equation $y'' + 9y = 0$; satisfying the initial conditions $y(0) = 0$ and $y'(0) = 2$. Given that

$$\mathscr{L}^{-1}\left[\frac{3}{s^2+9}\right] = \sin 3t$$

3

(c) Evaluate the function $F(t)$, whose inverse Laplace transform is

$$f(s) = \log\left(\frac{s^2-1}{s^2}\right)$$

3

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