Student's Copy

2023

(CBCS)

(5th Semester)

MATHEMATICS

SEVENTH PAPER

(Complex Analysis)

Full Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

(SECTION : A—OBJECTIVE)

(*Marks* : 10)

Tick ☑ the correct answer in the boxes provided : 1×10=10

1. Which one of the following is correct for any two complex numbers z_1 and z_2?

   (a) $||z_1|-|z_2|| \leq (|z_1|-|z_2|)$ ☐

   (b) $||z_1|-|z_2|| \leq (|z_1|+|z_2|)$ ☐

   (c) $||z_1|+|z_2|| \geq (|z_1|+|z_2|)$ ☐

   (d) $||z_1|-|z_2|| \geq (|z_1|-|z_2|)$ ☐

2. The complex number z satisfying $|z-1|=|z-3|=|z-i|$ is

(a) $2+i$ ☐

(b) $\frac{3}{2}+\frac{1}{2}i$ ☐

(c) $2+2i$ ☐

(d) None of the above ☐

3. The value of z for which the function w defined by $z=e^{-v}(\cos u+i\sin u)$, where $w=u+iv$ ceases to be analytic, is

(a) $z=0$ ☐

(b) $z=\infty$ ☐

(c) $z=i$ ☐

(d) $z=-i$ ☐

4. Harmonic conjugate of the function $(x-1)^3-3xy^2+3y^2$ is

(a) $3(x^2-y^2)+c$ ☐

(b) $6y(1-x)+c$ ☐

(c) $3x^2y-6xy+3y-y^3+c$ ☐

(d) $3(x-1)^2-3xy+6y+c$ ☐

5. The series $\Sigma n!z^n$ is convergent only for the value of

(a) $z\neq 0$ ☐

(b) $0<|z|<1$ ☐

(c) $1<|z|$ ☐

(d) $z=0$ ☐

6. The radius of convergence of the power series $\Sigma z^{n!}$ is

(a) 0 ☐

(b) ∞ ☐

(c) 1 ☐

(d) a real number greater than 1 ☐

7. The value of $\int_C \text{Re}(z)\, dz$, $C : |z| = 1$, is

(a) 0 ☐

(b) 2π ☐

(c) πi ☐

(d) $2\pi i$ ☐

8. The value of $\int_C z^n\, dz$, $C : |z| = r$ and $n \neq -1$, is

(a) 0 ☐

(b) 2π ☐

(c) $2\pi i$ ☐

(d) 1 ☐

9. The function $f(z) = \log z$ has a/an

(a) isolated singularity at $z = 0$ ☐

(b) non-isolated singularity at $z \neq 0$ ☐

(c) essential singularity at $z = 0$ ☐

(d) isolated essential singularity at $z = 0$ ☐

10. The limit point of the poles of a function $f(z)$ is

(a) a pole ☐

(b) a non-isolated singularity ☐

(c) an isolated singularity ☐

(d) a non-isolated essential singularity ☐

(SECTION : B—SHORT ANSWERS)

(*Marks* : 15)

Answer the following :

3×5=15

UNIT—I

1. For what value of k does the equation $z\bar{z} + (-3+4i)\bar{z} - (3+4i)z + k = 0$ represent a circle?

OR

2. For any two complex numbers z_1 and z_2, prove that $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$ if and only if $z_1\bar{z}_2$ is purely imaginary.

UNIT—II

3. For analytic function $f(z)$, prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$

OR

4. If $u - v = (x-y)(x^2 + 4xy + y^2)$ and $f(z) = u + iv$ is an analytic function of $z = x + iy$, then find $f(z)$ in terms of z.

UNIT—III

5. Show that the derivative series of the power series $\Sigma a_n z^n$ has the same radius of convergence as the original series.

OR

6. Find the radius of convergence of the series

$$\Sigma \frac{\left(1+\frac{1}{n}\right)^{n^2}}{n^3} z^n$$

UNIT—IV

7. Evaluate :

$$\int_{-2+i}^{5+3i} z^3 \, dz$$

OR

8. If a function $f(z)$ is analytic for finite value of z and is bounded, then show that $f(z)$ is constant.

UNIT—V

9. Define a pole of order n with example.

OR

10. What kind of singularity does the function $f(z) = \dfrac{1}{1-e^z}$ have at $z = 2\pi i$?

(SECTION : C—DESCRIPTIVE)

(*Marks* : 50)

Answer the following : 10×5=50

UNIT—I

1. *(a)* Find the locus of the point z satisfying $|z-1|+|z+1| \le 4$. 5

(b) Prove that the two points z_1 and z_2 will be inverse point with respect to the circle $z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + r = 0$ if and only if $z_1\bar{z}_2 + \bar{\alpha}z_1 + \alpha\bar{z}_2 + r = 0$. 5

OR

2. *(a)* Determine the region of Argand diagram determined by

$$\left|\frac{z-1}{z+1}\right| \le 2$$ 5

(b) Find the inverse of the point $(1+i)$ with respect to the circle whose centre is at i and radius is 2. 5

UNIT—II

3. (a) Prove that the function $f(z) = u(x, y) + iv(x, y)$ is analytic in the domain D if the partial derivatives u_x, u_y, v_x, v_y exist, are continuous and satisfy Cauchy-Riemann equation at each point of D.

(b) Show that the function $\frac{1}{z^4}$, $z \neq 0$ is analytic and find its derivative.

OR

4. (a) Show that the derivative of $w = f(z)$ for $z = re^{i\theta}$ is $\frac{\partial w}{\partial r} e^{-i\theta}$.

(b) Examine the analyticity of the function

$$f(z) = \frac{x^2 y^5 (x + iy)}{x^4 + y^{10}}, \quad z \neq 0; \quad f(0) = 0$$

in a region including origin.

UNIT—III

5. (a) State and prove Cauchy-Hadamard theorem.

(b) Find the radius of convergence of the power series $\sum \frac{in + 2}{2^n} z^n$.

OR

6. (a) Find the centre and radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{1.3.5 \ldots (2n - 1)}{n!} \left(\frac{1 - z}{z} \right)^n$$

(b) Examine the behaviour of the power series $\sum_{n=1}^{\infty} \frac{z^{4n}}{4n + 1}$ on the circle of convergence.

UNIT—IV

7. (a) State and prove Cauchy integral formula for an analytical function.

(b) By using Cauchy integral formula, evaluate $\int_C \frac{dz}{z(z + \pi i)}$, where C is $|z + 3i| = 4$.

OR

8. *(a)* State and prove Taylor's theorem. 5

(b) Expand the function $\dfrac{z^2 - 1}{(z+2)(z+3)}$ in the following regions : 5

(i) $|z| < 2$

(ii) $2 < |z| < 3$

UNIT—V

9. *(a)* Prove that a function $f(z)$, which is regular everywhere except at infinity where it has a pole of order n, is a polynomial of degree n. 5

(b) Check all the singularities of

$$f(z) = \frac{1}{\tan\left(\dfrac{\pi}{z}\right)}$$

which lie on the real axis from $z = -1$ to $z = 1$. 5

OR

10. Find the kinds of singularity of the following functions : 10

(i) $\tan\left(\dfrac{1}{z}\right)$ at $z = 0$

(ii) $\sin\left(\dfrac{1}{1-z}\right)$ at $z = 1$

(iii) $\sin z - \cos z$ at $z = \infty$

(iv) $\dfrac{z^2 + 4}{e^z}$ at $z = \infty$

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