MATH/V/CC/06 **Student's Copy**

2023

(CBCS)

(5th Semester)

MATHEMATICS

SIXTH PAPER

(Real Analysis)

Full Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

(SECTION : A—OBJECTIVE)

(*Marks* : 10)

Tick ☑ the correct answer in the boxes provided : 1×10=10

1. Every finite set is

(a) open but not closed ☐

(b) both closed and open ☐

(c) neither open nor closed ☐

(d) closed but not open ☐

2. Let ξ be a limit point of a set S and N be any neighbourhood of ξ, then

(a) $N \cap S = \phi$ ☐

(b) $N \cap S = \{\xi\}$ ☐

(c) $N \cap S$ is a finite set ☐

(d) $N \cap S$ is an infinite set ☐

3. Every subset of the discrete metric space is

(a) neither open nor closed ☐

(b) open as well as closed ☐

(c) open but not closed ☐

(d) closed but not open ☐

4. Which of the following metric spaces is not compact?

(a) A closed interval with the usual metric ☐

(b) The discrete space (X, d), where X is a finite set ☐

(c) The space (R, d), where R is the set of real numbers with the usual metric ☐

(d) A closed subset of a compact metric space ☐

5. A real-valued function continuous on a compact set is

(a) bounded above but not below ☐

(b) bounded and attains its bounds ☐

(c) bounded below but not above ☐

(d) bounded and does not attain its bounds ☐

6. A function $f : D \to R$, $D \subset R^n$ is continuous if and only if

(a) $f^{-1}(V)$ is closed in R for every closed set V in D ☐

(b) $f(V)$ is closed in R for every closed set V in D ☐

(c) $f^{-1}(V)$ is open in R for any set V in D ☐

(d) $f(V)$ is open in R for every open set V in R ☐

7. Let $f : D \to R$, $D \subset R^2$ is differentiable at a point $(x, y) \in D$, if for any point $(x+h, y+k)$ in a neighbourhood of (a, b), the change in f can be expressed as

(a) $f_x(x, y)k + f_y(x, y)h + h\phi + k\psi$, where ϕ and ψ are functions of h, k and both tend to zero as $(h, k) \to (0, 0)$ ☐

(b) $f_x^2(x, y)h + f_y^2(x, y)k + h\phi + k\psi$, where ϕ and ψ are functions of h, k and both tend to zero as $(h, k) \to (0, 0)$ ☐

(c) $f_x(x, y)h + f_y(x, y)k + h\phi + k\psi$, where ϕ and ψ are functions of h, k and both tend to zero as $(h, k) \to (0, 0)$ ☐

(d) $f_x^2(x, y)k + f_y^2(x, y)h + h\phi + k\psi$, where ϕ and ψ are functions of h, k and both tend to zero as $(h, k) \to (0, 0)$ ☐

8. If $y_1, y_2, ..., y_n$ are determined as functions of $x_1, x_2, ..., x_n$ by the functional equations $f_i(x_1, x_2, ..., x_n; y_1, y_2, ..., y_n) = 0$, $i = 1$ to n, then $\frac{\partial(f_1, f_2, ..., f_n)}{\partial(x_1, x_2, ..., x_n)} =$

(a) $\frac{\partial(f_1, f_2, ..., f_n)}{\partial(y_1, y_2, ..., y_n)} \frac{\partial(y_1, y_2, ..., y_n)}{\partial(x_1, x_2, ..., x_n)}$ ☐

(b) $-\frac{\partial(f_1, f_2, ..., f_n)}{\partial(y_1, y_2, ..., y_n)} \frac{\partial(y_1, y_2, ..., y_n)}{\partial(x_1, x_2, ..., x_n)}$ ☐

(c) $(-1)^n \frac{\partial(f_1, f_2, ..., f_n)}{\partial(y_1, y_2, ..., y_n)} \frac{\partial(y_1, y_2, ..., y_n)}{\partial(x_1, x_2, ..., x_n)}$ ☐

(d) $(-1)^{n+1} \frac{\partial(f_1, f_2, ..., f_n)}{\partial(y_1, y_2, ..., y_n)} \frac{\partial(y_1, y_2, ..., y_n)}{\partial(x_1, x_2, ..., x_n)}$ ☐

9. Let $f : D \to R$, $D \subset R^n$ and $(a, b) \in D$. Then $f_{xy}(a, b) = f_{yx}(a, b)$ if

(a) f_x and f_y are both continuous at (a, b) ☐

(b) f_x is continuous at (a, b) and f_y exists at (a, b) ☐

(c) f_x and f_y are both differentiable at (a, b) ☐

(d) f_x is differentiable at (a, b) and f_y exists at (a, b) ☐

10. A necessary condition for $f(x, y)$ to have an extreme value at (a, b) is that

(a) $f_x(a, b) = 0 = f_y(a, b)$ ☐

(b) $f_{xx}(a, b) = 0 = f_{yy}(a, b)$ ☐

(c) $f_{xy}(a, b) = 0 = f_{yx}(a, b)$ ☐

(d) $f_x(a, b) = f_y(0, 0) \neq 0$ ☐

(SECTION : B—SHORT ANSWERS)

(*Marks* : 15)

Answer the following : $3 \times 5 = 15$

UNIT—I

1. Show that every open set is a union of open intervals.

OR

2. Show by an example that boundedness is not a necessary condition for an infinite set to have a limit point.

UNIT—II

3. Let (X, d) be any metric space. Prove that a subset of X is closed if and only if its complement is open.

OR

4. Show that the discrete space is a complete metric space.

UNIT—III

5. Prove that the image of a compact set under a continuous function is compact.

OR

6. Show that the function

$$f(x, y) = \begin{cases} \dfrac{2x(x^2 - y^2)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is continuous at $(0, 0)$.

UNIT—IV

7. Let $f : D \to R, D \subset R^2$ and $(a, b) \in D$. If f_x exists throughout a neighbourhood of a point of (a, b) and $f_y(a, b)$ exists, then prove that for any point $(a + h, b + k)$ of this neighbourhood

$$f(a + h, b + k) - f(a, b) = hf_x(a + \theta h, b + k) + k\{f_y(a, b) + \eta\}$$

where $0 < \theta < 1$ and η is a function of k which tends to 0 with k.

OR

8. Show that the function

$$f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is differentiable at the origin.

UNIT—V

9. Show that $f_{xy}(0, 0) = f_{yx}(0, 0)$ for the function

$$f(x, y) = \begin{cases} \dfrac{(x^2 y^2)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

OR

10. Using Taylor's theorem, expand $e^x \cos(y)$ about $\left(0, \frac{\pi}{2}\right)$ up to the second degree term.

(SECTION : C—DESCRIPTIVE)

(*Marks* : 50)

10×5=50

Answer the following :

UNIT—I

1. (a) State and prove Bolzano-Weierstrass theorem. 1+5=6

(b) Prove that the derived set of a set is closed. 4

OR

2. (a) State and prove Heine-Borel theorem using Lindelöf covering theorem. 1+5=6

(b) If a sequence of closed intervals $[a_n, b_n]$ is such that $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ and $\lim_{n\to\infty}(a_n - b_n) = 0$, then prove that there is one and only one point common to all the intervals. 4

UNIT—II

3. (a) Prove that the space R^n of all ordered n-tuples with the metric d, where

$$d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}}$$

is a complete metric space. 6

(b) Prove that every closed subset of a compact metric space is compact. 4

OR

4. (a) Prove that any compact subset of a metric space is closed and bounded. 5

(b) Let (X, d) be a complete metric space. Let Y be any non-empty subset of X. Then show that Y is complete if and only if it is closed. 5

UNIT—III

5. (a) Define uniform continuity of a function of n-variables. Prove that a function continuous on a compact set is uniformly continuous. 1+5=6

(b) Discuss the continuity of the function $f(x, y) = \sqrt{|xy|}$ at $(0, 0)$. 4

OR

6. (a) Prove that a function $f: D \to R$, $D \subset R^n$ is continuous if and only if $f^{-1}(U)$ is open in R^n for every open set U in R. 5

(b) Let $f: D \to R$, $D \subset R^n$, where D is a convex set. Show that f assumes every value between $f(x)$ and $f(y)$, $\forall\, x, y \in D$. 5

UNIT—IV

7. (a) If α, β, γ are the roots of the equation in t such that

$$\frac{u}{a+t} + \frac{v}{b+t} + \frac{w}{c+t} = 1$$

then show that

$$\frac{\partial(u, v, w)}{\partial(\alpha, \beta, \gamma)} = \frac{(\beta - \alpha)(\beta - \gamma)(\gamma - \alpha)}{(a-b)(b-c)(c-a)}$$

5

(b) Let $f : D \to R$, $D \subset R^2$ and $(a, b) \in D$. Show that if f_x is continuous at (a, b) and $f_y(a, b)$ exists, then f is differentiable at (a, b). 5

OR

8. (a) Prove that a function which is differentiable at a point possesses the first-order partial derivatives at that point but the converse is not necessarily true. 6

(b) Find the directional derivative of $f(x, y) = x\cos y$ in the direction of $\vec{v} = (2, 1)$ at the point $(0, \pi)$. 4

UNIT—V

9. (a) State and prove Schwarz's theorem for a function of two variables. 1+5=6

(b) Show that the condition of Young's theorem is not satisfied for the function

$$f(x, y) = \begin{cases} (x^2 + y^2)\log(x^2 + y^2), & \text{when } x^2 + y^2 \neq 0 \\ 0, & \text{when } x = y = 0 \end{cases}$$

at $(0, 0)$. 4

OR

10. (a) State Taylor's theorem for two variables. Hence, find the expansion of $e^{ax}\cos(by)$ up to four terms. 1+4=5

(b) Examine any *one* of the following functions for extreme values : 5

(i) $f(x, y) = (x^2 + y^2 - 4)^2 - x^2$

(ii) $f(x, y) = x^2y^2 - 5x^2 - 8xy - 5y^2$

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