(SECTION : B—SHORT ANSWERS)

(*Marks* : 15)

Answer the following : $3 \times 5 = 15$

UNIT—I

1. Solve $\frac{dy}{dx} = \sin(x+y)$.

OR

2. Check the exactness of the differential equation

$$\left\{y\left(1+\frac{1}{x}\right)+\cos y\right\}dx + (x + \log x - x\sin x)\,dy = 0$$

and solve it.

UNIT—II

3. Solve $(D^2 + 5 - 2D)y = 10\sin x$.

OR

4. Find the particular integral of $(D^3 + D^2 - 6D)y = 1 + x^2$, where $D = \frac{dy}{dx}$.

UNIT—III

5. Find the general and singular solutions of $p^3 + y^2 = 1$.

OR

6. Solve $p + \frac{1}{p} = \frac{10}{3}$.

UNIT—IV

7. Solve $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 2\log x.$

OR

8. Solve $(yz + xyz)dx + (zx + xyz)dy + (xy + xyz)dz = 0.$

UNIT—V

9. Find the general solution of $xzp + yzq = xy$.

OR

10. Find the integral of $p^3 + q^3 = 27z.$

(SECTION : C—DESCRIPTIVE)

(*Marks* : 50)

Answer the following : 10×5=50

UNIT—I

1. (a) Find the differential equation of the family of curves $y = Ae^{3x} + Be^{5x}$ for different values of A and B. 5

(b) Solve the differential equation

$$(x + y)^2 \frac{dy}{dx} = a^2$$ 5

OR

2. (a) Solve :

$$(1 - x^2)\frac{dy}{dx} - xy = 1$$ 5

(b) Reduce the equation $(3x^2 + 4xy)\,dx + (2x^2 + 2y)\,dy = 0$ to exact form and solve it. 5

UNIT—II

3. Solve the following : 5+5=10

(a) $(D^2 - 2D + 1)y = x \sin x$

(b) $(D^3 + 3D^2 + 2D)y = x^2$

OR

4. (a) Solve $(D^2 - 2D + 4)y = e^x \cos x$. 5

(b) Solve the differential equation

$$(D^2 + 4)y = x \sin x$$

5

UNIT—III

5. (a) Solve the differential equation

$$p^3 - p(y+3) + x = 0$$

where $p = \frac{dy}{dx}$. 5

(b) Find the orthogonal trajectories of the family of curves

$$\frac{x^2}{a^2} + \frac{y^2}{b^2 + \lambda} = 1$$

λ being parameter. 5

OR

6. (a) Reduce the differential equation $(px - y)(x - yp) = 2p$ to Clairaut's form by substituting $x^2 = u$, $y^2 = v$ and find its general and singular solutions. 5

(b) Solve $y = x + p^3$, where $p = \frac{dy}{dx}$. 5

UNIT—IV

7. (a) Solve the equation

$$x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$$

5

(b) Show that the equation

$$x\frac{d^3y}{dx^3} + (x^2 - 3)\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = 0$$

is exact and solve it. 5

OR

8. (a) Check the condition of integrability and solve the equation

$$yz^2(x^2 - yz)dx + zx^2(y^2 - xz)dy + xy^2(z^2 - xy)dz = 0$$ 5

(b) Transform the equation

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} + 4x^3y = x^5$$

and find the solution. 5

UNIT—V

9. (a) Use Lagrange's auxiliary equations to solve the equation

$$(x^3 + 3xy^2)p + (y^3 + 3x^2y)q = 2(x^2 + y^2)z$$ 5

(b) Apply Charpit's method to find the complete integral of the equation $px + qy = pq$. 5

OR

10. (a) Find the general solution of the equation

$$(y^2 + z^2 - x^2)p - 2xyq + 2zx = 0$$ 5

(b) Solve $\cos(x + y)p + \sin(x + y)q = z$. 5

★★★

MATH/V/CC/05 **Student's Copy**

2023

(CBCS)

(5th Semester)

MATHEMATICS

FIFTH PAPER

(Computer-Oriented Numerical Analysis)

Full Marks : 75

Time : 3 hours

Use of Calculator is allowed

(SECTION : A—OBJECTIVE)

(*Marks* : 10)

Each question carries 1 mark

Put a Tick ☑ mark against the correct answer in the boxes provided :

1. Δe^{ax+b} (where a and b are constants and Δ denotes the forward difference operator) equals to

   (a) $e^{ax+b}(e^h+1)$ ☐

   (b) $e^{ax+b}(e^h-1)$ ☐

   (c) $e^{ax+b}(e^{ah}+1)$ ☐

   (d) $e^{ax+b}(e^{ah}-1)$ ☐

2. The second approximation of the root of the equation $x^3 - x - 1 = 0$ using bisection method is

(a) 1·5 ☐ (b) 1·25 ☐

(c) 2·5 ☐ (d) 2·25 ☐

3. If $f(x) = 1/x$, then the divided difference of $\delta(b, a)$ is

(a) $-\frac{1}{ab}$ ☐ (b) $\frac{1}{ab}$ ☐

(c) $\frac{(a+b)}{ab}$ ☐ (d) $\frac{(a-b)}{ab}$ ☐

4. Lagrange's interpolation polynomial for the data $f(0) = 0$, $f(1) = 0$ and $f(3) = 6$ is

(a) $(x-1)(x+1)$ ☐

(b) $x^2 + 1$ ☐

(c) $x(x-1)$ ☐

(d) $x(x+1)$ ☐

5. The method of obtaining the solution of the system of equations by reducing the matrix A to a diagonal matrix is known as

(a) Gauss-Seidel iteration method ☐

(b) Gauss elimination method ☐

(c) Gauss-Jordan method ☐

(d) Crout's method ☐

6. The system of equations

$$a_1x + a_2y + a_3z = d_1;\ b_1x + b_2y + b_3z = d_2;\ c_1x + c_2y + c_3z = d_3$$

is a diagonal system, if

(a) $|a_1| \ge |a_2| + |a_3|,\ |b_1| \ge |b_2| + |b_3|,\ |c_1| \ge |c_2| + |c_3|$ ☐

(b) $|a_1| \ge |a_2| + |a_3|,\ |b_2| \ge |b_1| + |b_3|,\ |c_3| \ge |c_1| + |c_2|$ ☐

(c) $|a_1| \le |a_2| + |a_3|,\ |b_1| \le |b_2| + |b_3|,\ |c_1| \le |c_2| + |c_3|$ ☐

(d) $|a_1| \le |a_2| + |a_3|,\ |b_2| \le |b_1| + |b_3|,\ |c_3| \le |c_1| + |c_2|$ ☐

7. When numerical integration is applied for the integration of a function of single variable, then the method is called

(a) trapezoidal rule ☐

(b) general quadrature ☐

(c) Simpson's 1/3rd rule ☐

(d) mechanical quadrature ☐

8. In the general quadrature formula $\int_{x_0}^{x_0+nh} y\,dx$ of Simpson's 1/3rd rule

(a) n must be even ☐

(b) n must be odd ☐

(c) $n = 1, 2, 3, \cdots$ ☐

(d) $n = 4$ and $n = 6$ ☐

9. For solving ordinary differential equation numerically, which among the following is applied if successive integration can be obtained easily?

(a) Euler's method ☐

(b) Taylor's method ☐

(c) Picard's method ☐

(d) Runge-Kutta method ☐

10. The Euler's modified formula is given by

(a) $y_{n+1} = y_n + hf(x_n, y_n)$ ☐

(b) $y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$ ☐

(c) $y_{n+1} = y_n + \frac{h}{3}[f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$ ☐

(d) $y_{n+1} = y_n + h[f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$ ☐

(SECTION : B—SHORT ANSWERS)

(*Marks* : 15)

Each question carries 3 marks

Answer the following :

UNIT—I

1. Prove that $E\nabla = \Delta = \delta E^{1/2}$, where Δ, ∇, δ and E are forward, backward, central and shift operator respectively.

OR

2. Find the second difference of the polynomial

$$f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9$$

with $h = 2$ and hence find $\Delta^4 f(x)$.

UNIT—II

3. Use Newton's forward interpolation formula to find the value of y for $x = 5$:

x	4	6	8	10
y	1	3	8	16

OR

4. Find the missing term in the given table :

x	2	3	4	5	6
y	45·0	49·2	54·1	—	67·4

UNIT—III

5. Solve the following system of equations by Crout's method :

$$x + y = 2 \text{ and } 2x + 3y = 5$$

OR

6. Solve the system of equations

$$2x + y = 3;\ 2x + 3y = 5$$

by Gauss-Seidel iteration method up to sixth iteration.

UNIT—IV

7. Find $f'(2)$ from the following table :

x	1	2	3	4	5
y	0	3	8	15	24

OR

8. Write the algorithm for Simpson's 1/3rd rule.

UNIT—V

9. Using Euler's method, solve $\frac{dy}{dx} = x - y$ with $y(0) = 1$ and find $y(0 \cdot 4)$ by taking $h = 0 \cdot 2$.

OR

10. Compute $y(0 \cdot 1)$ by Runge-Kutta method of fourth-order for the differential equation

$$\frac{dy}{dx} = xy + y^2, \; y(0) = 1$$

(SECTION : C—DESCRIPTIVE)

(*Marks* : 50)

The figures in the margin indicate full marks for the questions

Answer the following : 10×5=50

UNIT—I

1. (a) Use the method of iteration to find the real root lying between 1 and 2 of the equation $x^3 - 3x + 1 = 0$. 6

   (b) Write an algorithm for solving a given equation by using bisection method. 4

OR

2. (a) Find the smallest positive root of $x^2 - x - 12 = 0$ by regula-falsi method. 5

   (b) Find the positive root of the equation

$$3x - \cos x - 1 = 0$$

   by Newton-Raphson method correct up to four decimal places. 5

UNIT—II

3. (a) Obtain Newton's forward interpolation formula for interpolation with equal intervals of the argument. 5

(b) Find the value of y at the point $x = 0 \cdot 045$ using the following table : 5

x	0·01	0·02	0·03	0·04	0·05
y	98·434	48·439	31·778	23·449	18·454

OR

4. (a) Deduce Lagrange's interpolation formula for unequal interval. 5

(b) Using Newton's divided difference formula, evaluate $f(8)$, given that

x	0	2	5	9	11
$f(x)$	1	5	116	712	1310

5

UNIT—III

5. (a) Solve the following by Gaussian elimination method : 3

$$3x + y + 2z = 3,\ 2x - 3y - z = -3,\ x + 2y + z = 4$$

(b) Solve the following by Crout's method : 7

$$2x - 6y + 8z = 24;\quad 5x + 4y - 3z = 2;\quad 3x + y + 2z = 16$$

OR

6. (a) Solve the following system of linear equations by Gauss-Jordan method : 4

$$x + y + z = 9,\quad 2x - 3y + 4z = 13,\quad 3x + 4y + 5z = 40$$

(b) Solve the following system of equations by Gauss-Seidel method : 6

$$6x + 15y + 2z = 72,\quad x + y + 54z = 110,\quad 27x + 6y - z = 85$$

UNIT—IV

7. (a) Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using trapezoidal rule with $h = 0 \cdot 2$. Hence determine the value of π. 4

(b) Obtain the first and second derivatives of the function tabulated below, at the point $x = 0 \cdot 51$: 6

x	0·4	0·5	0·6	0·7	0·8
y	1·5836494	1·7974426	2·0442376	2·3275054	2·6510818

OR

8. (a) Obtain the formula for the second-order derivative $\frac{d^2y}{dx^2}$ or $f''(x)$ for numerical differentiation. 5

(b) Compute by Simpson's rule the value of the integral taking 10 points : 5

$$I = \int_0^5 e^{-x^2} dx$$

UNIT—V

9. (a) Solve $y' = x - y^2$, $y(0) = 1$ at the point 0·1 up to four decimal places by Taylor's series method. 5

(b) Find the value of $y(0 \cdot 1)$ by Picard's method given that

$$\frac{dy}{dx} = \frac{y - x}{y + x}, \quad y(0) = 1$$ 5

OR

10. (a) Apply Euler's method with $h = 0 \cdot 1$ to find the solution of the equation $y' = x + y$ with initial condition $y = 1$ for $x = 0$ in the range $0 \le x \le 1$. 5

(b) Given

$$\frac{dy}{dx} = \frac{1}{x + y}$$

$y(0) = 2$. If $y(0 \cdot 2) = 2 \cdot 09$, $y(0 \cdot 4) = 2 \cdot 17$ and $y(0 \cdot 6) = 2 \cdot 24$, then find $y(0 \cdot 8)$ using Milne's method. 5

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2 0 2 3

(CBCS)

(5th Semester)

MATHEMATICS

FIFTH PAPER

(Computer-Oriented Numerical Analysis)

Full Marks : 75

Time : 3 hours

Use of Calculator is allowed

(SECTION : A—OBJECTIVE)

(*Marks* : 10)

Each question carries 1 mark

Put a Tick ☑ mark against the correct answer in the boxes provided :

1. Δe^{ax+b} (where a and b are constants and Δ denotes the forward difference operator) equals to

   (a) $e^{ax+b}(e^h + 1)$ ☐

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2. The second approximation of the root of the equation $x^3 - x - 1 = 0$ using bisection method is

(a) 1·5 ☐ (b) 1·25 ☐

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(SECTION : C—DESCRIPTIVE)

(*Marks* : 50)

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$$\frac{dy}{dx} = \frac{y - x}{y + x}, \quad y(0) = 1$$ 5

OR

10. (a) Apply Euler's method with $h = 0 \cdot 1$ to find the solution of the equation $y' = x + y$ with initial condition $y = 1$ for $x = 0$ in the range $0 \le x \le 1$. 5

(b) Given

$$\frac{dy}{dx} = \frac{1}{x + y}$$

$y(0) = 2$. If $y(0 \cdot 2) = 2 \cdot 09$, $y(0 \cdot 4) = 2 \cdot 17$ and $y(0 \cdot 6) = 2 \cdot 24$, then find $y(0 \cdot 8)$ using Milne's method. 5

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2023

(CBCS)

(5th Semester)

MATHEMATICS

SIXTH PAPER

(Real Analysis)

Full Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

(SECTION : A—OBJECTIVE)

(*Marks* : 10)

Tick ☑ the correct answer in the boxes provided : 1×10=10

1. Every finite set is

(a) open but not closed ☐

(b) both closed and open ☐

(c) neither open nor closed ☐

(d) closed but not open ☐

2. Let ξ be a limit point of a set S and N be any neighbourhood of ξ, then

(a) $N \cap S = \phi$ ☐

(b) $N \cap S = \{\xi\}$ ☐

(c) $N \cap S$ is a finite set ☐

(d) $N \cap S$ is an infinite set ☐

3. Every subset of the discrete metric space is

(a) neither open nor closed ☐

(b) open as well as closed ☐

(c) open but not closed ☐

(d) closed but not open ☐

4. Which of the following metric spaces is not compact?

(a) A closed interval with the usual metric ☐

(b) The discrete space (X, d), where X is a finite set ☐

(c) The space (R, d), where R is the set of real numbers with the usual metric ☐

(d) A closed subset of a compact metric space ☐

5. A real-valued function continuous on a compact set is

(a) bounded above but not below ☐

(b) bounded and attains its bounds ☐

(c) bounded below but not above ☐

(d) bounded and does not attain its bounds ☐

6. A function $f: D \to R$, $D \subset R^n$ is continuous if and only if

(a) $f^{-1}(V)$ is closed in R for every closed set V in D ☐

(b) $f(V)$ is closed in R for every closed set V in D ☐

(c) $f^{-1}(V)$ is open in R for any set V in D ☐

(d) $f(V)$ is open in R for every open set V in R ☐

7. Let $f: D \to R$, $D \subset R^2$ is differentiable at a point $(x, y) \in D$, if for any point $(x+h, y+k)$ in a neighbourhood of (a, b), the change in f can be expressed as

(a) $f_x(x, y)k + f_y(x, y)h + h\phi + k\psi$, where ϕ and ψ are functions of h, k and both tend to zero as $(h, k) \to (0, 0)$ ☐

(b) $f_x^2(x, y)h + f_y^2(x, y)k + h\phi + k\psi$, where ϕ and ψ are functions of h, k and both tend to zero as $(h, k) \to (0, 0)$ ☐

(c) $f_x(x, y)h + f_y(x, y)k + h\phi + k\psi$, where ϕ and ψ are functions of h, k and both tend to zero as $(h, k) \to (0, 0)$ ☐

(d) $f_x^2(x, y)k + f_y^2(x, y)h + h\phi + k\psi$, where ϕ and ψ are functions of h, k and both tend to zero as $(h, k) \to (0, 0)$ ☐

8. If $y_1, y_2, \ldots, y_n$ are determined as functions of $x_1, x_2, \ldots, x_n$ by the functional equations $f_i(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n) = 0$, $i = 1$ to n, then $\frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(x_1, x_2, \ldots, x_n)} =$

(a) $\frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(y_1, y_2, \ldots, y_n)} \frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)}$ ☐

(b) $-\frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(y_1, y_2, \ldots, y_n)} \frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)}$ ☐

(c) $(-1)^n \frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(y_1, y_2, \ldots, y_n)} \frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)}$ ☐

(d) $(-1)^{n+1} \frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(y_1, y_2, \ldots, y_n)} \frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)}$ ☐

9. Let $f : D \to R$, $D \subset R^n$ and $(a, b) \in D$. Then $f_{xy}(a, b) = f_{yx}(a, b)$ if

(a) f_x and f_y are both continuous at (a, b) ☐

(b) f_x is continuous at (a, b) and f_y exists at (a, b) ☐

(c) f_x and f_y are both differentiable at (a, b) ☐

(d) f_x is differentiable at (a, b) and f_y exists at (a, b) ☐

10. A necessary condition for $f(x, y)$ to have an extreme value at (a, b) is that

(a) $f_x(a, b) = 0 = f_y(a, b)$ ☐

(b) $f_{xx}(a, b) = 0 = f_{yy}(a, b)$ ☐

(c) $f_{xy}(a, b) = 0 = f_{yx}(a, b)$ ☐

(d) $f_x(a, b) = f_y(0, 0) \neq 0$ ☐

(SECTION : B—SHORT ANSWERS)

(*Marks* : 15)

Answer the following : $3\times5=15$

UNIT—I

1. Show that every open set is a union of open intervals.

OR

2. Show by an example that boundedness is not a necessary condition for an infinite set to have a limit point.

UNIT—II

3. Let (X, d) be any metric space. Prove that a subset of X is closed if and only if its complement is open.

OR

4. Show that the discrete space is a complete metric space.

UNIT—III

5. Prove that the image of a compact set under a continuous function is compact.

OR

6. Show that the function

$$f(x, y) = \begin{cases} \dfrac{2x(x^2 - y^2)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is continuous at (0, 0).

UNIT—IV

7. Let $f : D \to R, D \subset R^2$ and $(a, b) \in D$. If f_x exists throughout a neighbourhood of a point of (a, b) and $f_y(a, b)$ exists, then prove that for any point $(a + h, b + k)$ of this neighbourhood

$$f(a + h, b + k) - f(a, b) = hf_x(a + \theta h, b + k) + k\{f_y(a, b) + \eta\}$$

where $0 < \theta < 1$ and η is a function of k which tends to 0 with k.

OR

8. Show that the function

$$f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2} & , \ (x, y) \neq (0, 0) \\ 0 & , \ (x, y) = (0, 0) \end{cases}$$

is differentiable at the origin.

UNIT—V

9. Show that $f_{xy}(0, 0) = f_{yx}(0, 0)$ for the function

$$f(x, y) = \begin{cases} \dfrac{(x^2 y^2)}{x^2 + y^2} & , \ (x, y) \neq (0, 0) \\ 0 & , \ (x, y) = (0, 0) \end{cases}$$

OR

10. Using Taylor's theorem, expand $e^x \cos(y)$ about $\left(0, \frac{\pi}{2}\right)$ up to the second degree term.

(SECTION : C—DESCRIPTIVE)

(*Marks* : 50)

Answer the following : 10×5=50

UNIT—I

1. *(a)* State and prove Bolzano-Weierstrass theorem. 1+5=6

(b) Prove that the derived set of a set is closed. 4

OR

2. *(a)* State and prove Heine-Borel theorem using Lindelöf covering theorem. 1+5=6

(b) If a sequence of closed intervals $[a_n, b_n]$ is such that $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ and $\lim_{n \to \infty} (a_n - b_n) = 0$, then prove that there is one and only one point common to all the intervals. 4

UNIT—II

3. (a) Prove that the space R^n of all ordered n-tuples with the metric d, where

$$d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}}$$

is a complete metric space. 6

(b) Prove that every closed subset of a compact metric space is compact. 4

OR

4. (a) Prove that any compact subset of a metric space is closed and bounded. 5

(b) Let (X, d) be a complete metric space. Let Y be any non-empty subset of X. Then show that Y is complete if and only if it is closed. 5

UNIT—III

5. (a) Define uniform continuity of a function of n-variables. Prove that a function continuous on a compact set is uniformly continuous. 1+5=6

(b) Discuss the continuity of the function $f(x, y) = \sqrt{|xy|}$ at $(0, 0)$. 4

OR

6. (a) Prove that a function $f : D \to R$, $D \subset R^n$ is continuous if and only if $f^{-1}(U)$ is open in R^n for every open set U in R. 5

(b) Let $f : D \to R$, $D \subset R^n$, where D is a convex set. Show that f assumes every value between $f(x)$ and $f(y)$, $\forall\, x, y \in D$. 5

UNIT—IV

7. (a) If α, β, γ are the roots of the equation in t such that

$$\frac{u}{a+t} + \frac{v}{b+t} + \frac{w}{c+t} = 1$$

then show that

$$\frac{\partial(u, v, w)}{\partial(\alpha, \beta, \gamma)} = \frac{(\beta - \alpha)(\beta - \gamma)(\gamma - \alpha)}{(a - b)(b - c)(c - a)}$$ 5

(b) Let $f: D \to R$, $D \subset R^2$ and $(a, b) \in D$. Show that if f_x is continuous at (a, b) and $f_y(a, b)$ exists, then f is differentiable at (a, b). 5

OR

8. (a) Prove that a function which is differentiable at a point possesses the first-order partial derivatives at that point but the converse is not necessarily true. 6

(b) Find the directional derivative of $f(x, y) = x \cos y$ in the direction of $\vec{v} = (2, 1)$ at the point $(0, \pi)$. 4

UNIT—V

9. (a) State and prove Schwarz's theorem for a function of two variables. 1+5=6

(b) Show that the condition of Young's theorem is not satisfied for the function

$$f(x, y) = \begin{cases} (x^2 + y^2)\log(x^2 + y^2), & \text{when } x^2 + y^2 \neq 0 \\ 0, & \text{when } x = y = 0 \end{cases}$$

at (0, 0). 4

OR

10. (a) State Taylor's theorem for two variables. Hence, find the expansion of $e^{ax} \cos(by)$ up to four terms. 1+4=5

(b) Examine any *one* of the following functions for extreme values : 5

(i) $f(x, y) = (x^2 + y^2 - 4)^2 - x^2$

(ii) $f(x, y) = x^2y^2 - 5x^2 - 8xy - 5y^2$

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