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(CBCS)

(5th Semester)

MATHEMATICS

SIXTH PAPER

(Real Analysis)

Full Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

(SECTION : A—OBJECTIVE)

(*Marks* : 10)

Tick ☑ the correct answer in the boxes provided : 1×10=10

1. Which of the following sets is a neighbourhood of each of its points?

(a) $\frac{1}{n} : n \quad N$ ☐

(b) The closed interval $[a, b]$ ☐

(c) The empty set ☐

(d) The set of integers ☐

2. Let   be a limit point of a set S and N be any neighbourhood of , then

(a) $N \quad S$ ☐

(b) $N \quad S$ is a finite set ☐

(c) $N \quad S \quad N \quad S$ ☐

(d) $N \quad S$ is an infinite set ☐

3. In the discrete space (X, d), for $a \quad X$, the open ball $B_r[a]$

(a) $\begin{array}{l} \{a\}, \text{ if } r \quad 1 \\ X\,, \text{ if } r \quad 1 \end{array}$ ☐

(b) $\begin{array}{l} \{a\}, \text{ if } 0 \quad r \quad 1 \\ X\,, \text{ if } \quad r \quad 1 \end{array}$ ☐

(c) $\begin{array}{l} \{a\}, \text{ if } 0 \quad r \quad 1 \\ X\,, \text{ if } \quad r \quad 1 \end{array}$ ☐

(d) $\begin{array}{l} \{a\}, \text{ if } 0 \quad r \quad 1 \\ X\,, \text{ if } \quad r \quad 1 \end{array}$ ☐

4. A closed subset of a complete metric space is

(a) compact and complete ☐

(b) compact but not complete ☐

(c) complete but not compact ☐

(d) neither compact nor complete ☐

5. A function $f : D \quad R,\ D \quad R^n$ is continuous if and only if

(a) $f \quad ^1(U)$ is closed in R^n for every closed set U in R □

(b) $f \quad ^1(U)$ is open in R^n for every closed set U in R □

(c) $f \quad ^1(U)$ is closed in R^n for every open set U in R □

(d) $f \quad ^1(U)$ is open in R^n for every set U in R □

6. Let $f : D \quad R,\ D \quad R^n$ be continuous and let $K \quad D$ be compact. Let $\{G \quad \}$ be an open cover of $f(K)$. Then $\{f \quad ^1(G \quad)\}$ is an open cover of

(a) D □

(b) $f(D)$ □

(c) K □

(d) $f(K)$ □

7. A set of n differentiable functions of n independent variables are functionally related

(a) if and only if the Jacobian of the function is zero □

(b) if and only if the Jacobian of the function is greater than 1 □

(c) if and only if the Jacobian of the function is negative □

(d) if and only if the Jacobian of the function is infinite □

8. The function

$$f(x, y) \quad \begin{cases} \dfrac{xy}{x^2 \quad 2y^2}, & (x, y) \quad 0 \\ 0\,, & (x, y) \quad 0 \end{cases}$$

is

(a) continuous at (0, 0) and both f_x and f_y exist at (0, 0) ☐

(b) discontinuous at (0, 0) and both f_x and f_y exist at (0, 0) ☐

(c) continuous at (0, 0) and neither f_x nor f_y exists at (0, 0) ☐

(d) discontinuous at (0, 0) and neither f_x nor f_y exists at (0, 0) ☐

9. The function $f(x, y) \quad |x| \quad |y|$

(a) has an extreme value at (0, 0) but the partial derivatives f_x and f_y do not exist at (0, 0) ☐

(b) has an extreme value at (0, 0) and both the partial derivatives f_x and f_y exist at (0, 0) ☐

(c) has no extreme value at (0, 0), even though both the partial derivatives f_x and f_y exist at (0, 0) ☐

(d) has no extreme value at (0, 0) and the partial derivatives f_x and f_y do not exist at (0, 0) ☐

10. The only stationary point of $f(x, y) \quad x^2 \quad y^2 \quad x \quad y \quad xy$ is

(a) $\frac{1}{3}, \quad \frac{1}{3}$ which is a point of minima ☐

(b) $\frac{1}{3}, \quad \frac{1}{3}$ which is a point of maxima ☐

(c) $\frac{1}{3}, \quad \frac{1}{3}$ which is a saddle point ☐

(d) $\frac{1}{3}, \frac{1}{3}$ which is a point of minima ☐

(SECTION : B—SHORT ANSWER)

(*Marks* : 15)

Answer the following : 3×5=15

UNIT—I

1. Prove that the interior of a set S is the largest open subset of S.

OR

2. Prove that the derived set of a set is closed.

UNIT—II

3. Prove that in a metric space, a set is closed if and only if its complement is open.

OR

4. Show that (X, d), where X is the set of real numbers and

$$d(x, y) \quad \left|\frac{x}{y}\right|, \quad x, y \quad R$$

is not a metric space.

UNIT—III

5. Examine the continuity of the function

$$f(x, y) \quad \begin{array}{ll} \dfrac{\sin\ ^{1}(x \quad 3y)}{\tan\ ^{1}(3x \quad 6y)}, & (x, y) \quad (0, 0) \\ 1 \qquad , & (x, y) \quad (0, 0) \end{array}$$

at the origin.

OR

6. Examine the continuity of the function

$$f(x, y) \quad \begin{array}{ll} \dfrac{x^2 \quad x\sqrt{y}}{x^2 \quad y}, & (x, y) \quad (0, 0) \\ 0 \qquad , & (x, y) \quad (0, 0) \end{array}$$

at the origin.

UNIT—IV

7. If $u \quad x \quad y \quad z, v \quad x \quad y \quad z, w \quad x^2 \quad y^2 \quad z^2 \quad 2yz,$ find

$$\frac{(u, v, w)}{(x, y, z)}$$

OR

8. Show by an example that a function which possesses the first-order partial derivatives at a point is not necessarily differentiable thereat.

UNIT—V

9. Prove that the function $f(x, y) \quad y^2 \quad x^2 \quad x^4$ has a minima at the origin.

OR

10. Show that the condition of Young's theorem is not satisfied for the function

$$f(x, y) \quad \begin{matrix} \dfrac{x^2y^2}{x^2 \quad y^2}, & \text{when} \quad (x, y) \quad (0, 0) \\ 0 \quad , & \text{when} \quad (x, y) \quad (0, 0) \end{matrix}$$

at (0, 0) even though $f_{xy}(0, 0) \quad f_{yx}(0, 0)$.

(SECTION : C—DESCRIPTIVE)

(*Marks* : 50)

Answer the following : 10×5=50

UNIT—I

1. *(a)* State and prove Bolzano-Weierstrass theorem for real line. 6

(b) Prove that the intersection of a finite number of open sets is open but this does not hold for infinite sets in general. 4

OR

2. *(a)* State and prove Heine-Borel theorem. 7

(b) Show that an infinite set with the discrete metric is not compact. 3

UNIT—II

3. *(a)* Prove that the space R^n of all ordered n-tuples with the metric d, where

$$d(x, y) = \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{\frac{1}{2}}$$

is a complete metric space. 6

(b) Show that every closed sphere in a metric space is closed. 4

OR

4. *(a)* Prove that every compact subset of a metric space is closed and bounded but the converse need not be true. 6

(b) Prove that every closed subset of a compact metric space is compact. 4

UNIT—III

5. *(a)* Define uniform continuity of a function in R^n. Prove that a function continuous on a compact set is uniformly continuous. 1+5=6

(b) Prove that the image of a compact set under a continuous function is compact. 4

OR

6. *(a)* Prove that a function $f : D \to R$, $D \subseteq R^n$ is continuous if and only if $f^{-1}(U)$ is open in R for every open set U in R. 5

(b) Let $f : D \to R$, $D \subseteq R^n$, where D is a convex set. Show that f assumes every value between $f(x)$ and $f(y)$, $\forall\, x, y \in D$. 5

UNIT—IV

7. *(a)* If α, β, γ are the roots of the equation in t such that

$$\frac{u}{a+t} + \frac{v}{b+t} + \frac{w}{c+t} = 1$$

then prove that

$$\frac{\partial(u, v, w)}{\partial(\alpha, \beta, \gamma)} = -\frac{(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)}{(a-b)(b-c)(c-a)}$$ 5

(b) If f_x exists throughout a neighbourhood of a point of (a, b) and $f_y(a, b)$ exists, then prove that for any point $(a+h, b+k)$ of this neighbourhood

$$f(a+h, b+k) - f(a, b) = hf_x(a+\theta h, b+k) + k\{f_y(a, b)+\eta\}$$

where $0<\theta<1$ and η is a function of k which tends to 0 with k. 5

OR

8. *(a)* Find the directional derivative of $f(x, y) = x^3 + y\sin x$ at the point $\left(0, \frac{\pi}{2}\right)$ in the direction of $\vec{u} = 3\hat{i} + 4\hat{j}$. 5

(b) Let $f : D \to R$, $D \subseteq R^2$ and $(a, b) \in D$. Show that if one of the partial derivatives exists and is bounded in a neighbourhood of (a, b) and the other exists at (a, b), then the function f is continuous at (a, b). 5

UNIT—V

9. *(a)* State and prove Young's theorem. 1+5=6

(b) Show that the conditions of Schwarz's theorem are not satisfied for the function

$$f(x, y) = \begin{cases} (x^2 + y^2)\log(x^2 + y^2), & x^2 + y^2 \neq 0 \\ 0 & , \; x = y = 0 \end{cases}$$

at (0, 0) and $f_{xy}(0, 0) = f_{yx}(0, 0)$. 4

OR

10. *(a)* Using Taylor's theorem, show that the expansion of $\sin(xy)$ in powers of $(x-1)$ and $\left(y-\frac{\pi}{2}\right)$ up to and including the second-degree term is

$$1 - \frac{\pi^2}{8}(x-1)^2 - \frac{\pi}{2}(x-1)\left(y-\frac{\pi}{2}\right) - \frac{1}{2}\left(y-\frac{\pi}{2}\right)^2$$

4

(b) Examine the following functions for extreme values : 3+3=6

(i) $f(x, y) = 4x^2 - xy + 4y^2 + x^3y + xy^3 - 4$

(ii) $f(x, y) = x^3 + y^3 - 3x^2 - 3y^2 + 9x$

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